Cosmology in Co-moving Coordinates

We disregard distances due to expansion or contraction and only consider co-moving distances. We set the scale factor to $a = 1$.

We begin by describing the topology of the universe. We set the norm of the following exponential equal to one.

$$||exp \begin{pmatrix} 0 & -i\chi & i\theta & -i\phi \\ i\chi & 0 & i\phi & i\theta \\ -i\theta & -i\phi & 0 & i\chi \\ i\phi & -i\theta & -i\chi & 0 \end{pmatrix} ||^2 = [\alpha\mathbf{1}+\beta(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k})][\alpha\mathbf{1}-\beta(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k})]$$

$$= \alpha^2 - \beta^2\chi^2 - \beta^2\theta^2 - \beta^2\phi^2 = 1$$

.

where $\alpha = \Sigma_{n=0}^{\infty}\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n)!}$ and $\beta = \Sigma_{n=0}^{\infty}\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}$.

For $\theta = \phi = 0$, the above reduces to $cosh^2\chi - sinh^2\chi = 1$ which shows that χ is unbounded.

The set $\{\mathbf{q} = \alpha\mathbf{1} + \beta(i\chi\mathbf{i} + i\theta\mathbf{j} + i\phi\mathbf{k}) : ||\mathbf{q}|| = 1\}$ constitutes a pseudo-spheroid we shall call S_P^3. It has the infinitesimal generators $\{i\mathbf{i}, i\mathbf{j}, i\mathbf{k}\}$.

For the following we use the generalized Robertson-Walker coordinates,

$$ds^2 = dt^2 - \left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2sin^2\theta d\phi^2\right]$$

Let $\gamma = \gamma(s)$ be the path of a photon through space-time parameterized by space-time arclength s. The scale factor is assumed to be $a = 1$.

Then

$$\frac{d\gamma}{ds} = \frac{dt}{ds}\partial_t + \frac{dr}{ds}\partial_r + \frac{d\theta}{ds}\partial_\theta + \frac{d\phi}{ds}\partial_\phi$$

Now, $\langle\partial_t, \partial_t\rangle = 1$, $\langle\partial_r, \partial_r\rangle = \frac{1}{1-kr^2}$, $\langle\partial_\theta, \partial_\theta\rangle = r^2$, and $\langle\partial_\phi, \partial_\phi\rangle = r^2sin^2\theta$

For a photon

$$0 = \langle \frac{d\gamma}{ds}, \frac{d\gamma}{ds} \rangle = \frac{dt^2}{ds^2} - \frac{1}{1-kr^2}\frac{dr^2}{ds^2} - r^2\frac{d\theta^2}{ds^2} - r^2 sin^2\theta \frac{d\phi^2}{ds^2}$$

and therefore

$$0 = dt^2 - \frac{dr^2}{1-kr^2} - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2$$

We are interested in paths terminating along a single line of sight so

$$d\theta = d\phi = 0$$

Then

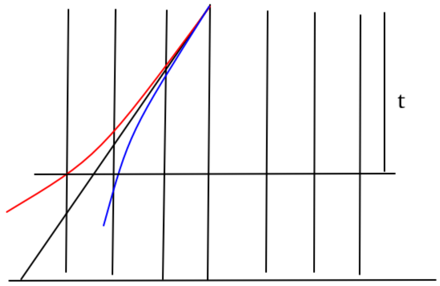
$$dt^2 = \frac{dr^2}{1-kr^2}$$

We have three cases:
For $k = 0$, $r = \rho$. For $k = +1$, $r = sin\beta$. For $k = -1$, $r = sinh\chi$

Then for the three cases, $dt = d\rho$, $dt = d\beta$, and $dt = d\chi$.

For flat space we get a cone as in special relativity. The other two cases are not cone-like but curved.

Assuming $\beta = \chi = 0$ when $\rho = 0$ we can say $\rho = \beta = \chi$. Then $sin\beta < \rho < sinh\chi$ and we have the following picture:



The above graph shows world-lines in co-moving co-ordinates. Also shown are the null paths followed by photons in the tri-curvature.

Red= $sinh\chi$; Black= ρ; and Blue= $sin\beta$

	$Density$	$SpectralShift$
Red	$Increased$	red
$Black$	$uniform$	$none$
$Blue$	$Decreased$	$blue$

In the above, *Density* refers to the observed density of galaxies per unit volume in the past. Spectral shift is due to time either moving slower or faster in the past. I.e. it depends on the ratio of proper time to t.

In light of the above, we propose the topology of the Universe to be $\mathbf{R}^+ \times S^3_P$ with metric $ds^2 = dt^2 - [d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$

We note above that $r = sinh\chi$ and so $\frac{dr}{dt} = cosh\chi\frac{d\chi}{dt} > 1$ for $\chi > 0$. Then the speed of light w.r.t. Robertson-Walker time is greater than 1. This is not unusual in Relativity Theory but it does not indicate that when c is measured locally it is different from its invariant value. With respect to proper time the speed of light always has its invariant value.

The speed of light is measured locally as its invariant value with respect to proper time τ as $\frac{dr}{d\tau} = 1$

For $k = 0$, $\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = \frac{d\rho}{dt}\frac{dt}{d\tau} = \frac{dt}{d\tau} = 1$ so $\frac{d\tau}{dt} = 1$

For $k = +1$, $\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = cos\beta\frac{dt}{d\tau} = 1$ so $cos\beta = \frac{d\tau}{dt} < 1$

For $k = -1$, $\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = cosh\chi\frac{dt}{d\tau} = 1$ so $cosh\chi = \frac{d\tau}{dt} > 1$

For $k = 0$ there is no spectral shift.

For $k = +1$, $\frac{d\tau}{dt} < 1$, so Robertson-Walker time t is running faster than proper time so the light source appears blue shifted.

For $k = -1$, $\frac{d\tau}{dt} > 1$, so Robertson-Walker time t is running slower than proper time so the light source appears red shifted. $z + 1 = cosh\chi$.

For the Cosmic Microwave Background, temperature at last scattering was $\approx 5000°K$ and is observed to be $\approx 2.7°K$. So, wavelengths have increased by a factor of 1850 and frequencies divided by the same amount.

Then $\frac{\lambda}{\lambda'} = \frac{\nu'}{\nu} = \frac{d\tau}{dt} \approx 1850 = cosh\chi$. Then $\chi = 8.21$.

World Lines:

Let O be an observer where $\chi = 0$. Assume the length of its world line to be t_0 which we can regard as the age of the Universe from the point of view of O.

Then the length of a world line at χ is $L = t_0\sqrt{cosh^2\chi + sinh^2\chi}$. This is the time when light emitted at O is observed at O' at χ. Though in co-moving coordinates the distances between world lines remains constant, the transit time for light is not. A photon leaving O at t_0 arrives at O' at

$$t_1 = t_0\sqrt{cosh^2\chi + sinh^2\chi}$$

having traversed the co-moving distance O to O' (at $\chi = X$) along the arc $(cosh\chi, sinh\chi)$ equal to (in units where $dt = d\chi$ at $\chi = 0$)

$$D = \int_0^X \sqrt{[\frac{dcosh(\chi)}{d\chi}]^2 + [\frac{dsinh(\chi)}{d\chi}]^2}d\chi$$

$$= \int_0^X \sqrt{sinh^2(\chi) + cosh^2(\chi)}d\chi$$

Define $D_0 = \int_0^1 \sqrt{sinh^2(\chi) + cosh^2(\chi)}d\chi$. This is our defining unit of the distance between world lines. At D_0, $z + 1 = cosh(1) = 1.54$.

Then $\int_0^2 \sqrt{sinh^2(\chi) + cosh^2(\chi)}d\chi \approx 3.2D_0$, $z + 1 = cosh(2) = 3.76$.

and $\int_0^3 \sqrt{sinh^2(\chi) + cosh^2(\chi)}d\chi \approx 10.45D_0$, $z + 1 = cosh(3) = 10.06$.

The ordinary rules of integration allow $\int_0^3 = \int_0^2 + \int_2^3$ but not $\int_0^3 = \int_0^2 + \int_0^1$. We emphasize $\int_0^3 \neq \int_0^2 + \int_0^1$. But we can say $\int_2^3 = \int_0^3 - \int_0^2$.

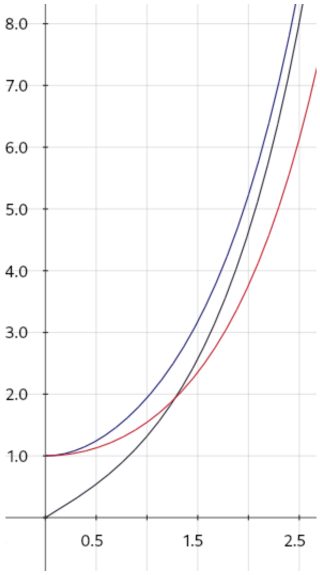
Assuming galaxies are uniformly distributed, the *observed* density of galaxies will equal $\frac{dD}{dX} = \sqrt{sinh^2(X) + cosh^2(X)}$.

$z + 1 = 2$ at $\chi \approx 1.31$ so $z = 1$ at that value.

Furthermore, using numerical methods shown below

$$\int_0^{1.31} \sqrt{sinh^2(\chi) + cosh^2(\chi)} d\chi \approx 2 \approx 1.5 D_0$$

The graph below shows the following in their corresponding colors: $cosh\chi$, $\int_0^X \sqrt{sinh^2(\chi) + cosh^2(\chi)} d\chi$, and $\sqrt{sinh^2(X) + cosh^2(X)}$.



Now consider light emitted at O' at t_1 and observed at O at t_0. Relative to O', O is at χ. Then $t_0 = t_1 \sqrt{cosh^2\chi + sinh^2\chi}$. It follows that $t_1 \to 0$ as $\chi \to \infty$. It follows that there is no $t = 0$ since χ has no upper bound and thus each world line exists on the interval $(0, \infty)$.

The Universe we are describing here in co-moving coordinates is infinite without any contraction or expansion (except for proper motion either way). It is uncaused without any beginning point in time.

We can also reintroduce scalar expansion (or contraction) on the topology

$$\mathbf{R}^+ \times S_P^3$$

with modified metric

$$ds^2 = dt^2 - a^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$$

where $a > 0$.

For a given $a > 0$, the arc $(acosh\chi, asinh\chi)$ traces out a length

$$D = a\int_0^X \sqrt{[\frac{dcosh(\chi)}{d\chi}]^2 + [\frac{dsinh(\chi)}{d\chi}]^2 d\chi}$$

$$= a\int_0^X \sqrt{sinh^2(\chi) + cosh^2(\chi)} d\chi$$

It is clear that for any X, $a\int_0^X \to 0$ as $a \to 0$. But this convergence is point-wise and not uniform since for any $a > 0$, $a\int_0^X \to \infty$ as $X \to \infty$. There is a singularity at $a = 0$ but due to the failure of uniform convergence, it is coordinate only and not physical.

We can change from big bang coordinates to co-moving by a transform of the time coordinate. $t = at'$ converts big bang coordinates to co-moving and $t' = at$ does the reverse. The scalar a satisfies the cosmological equations:

$$\begin{aligned} \frac{8\pi G\rho}{3} &= \frac{k}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 && (1) \\ 8\pi Gp &= -2\frac{\ddot{a}}{a} - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 && (2) \end{aligned}$$

where $k = -1$, ρ is the mass-energy density, and p is the cosmological pressure.

Letting $\eta = \int \frac{dt}{a}$ we can also say $dt \to d\eta$ transforms big bang coordinates to co-moving and $d\eta \to dt$ does the reverse.

A singularity which can be removed by a coordinate transformation is not physically real.

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