Dark Matter, Dark Energy and Critical Density

Summary: This article examines the question of critical density and its requirement of dark matter and dark energy. We propose an alternative.

Let us begin with the cosmological equations in their original form without a cosmological constant Λ.

$$\begin{aligned} \frac{8\pi G\rho}{3} &= \frac{k}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 && (1) \\ 8\pi G p &= -2\frac{\ddot{a}}{a} - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 && (2) \\ \frac{8\pi G}{3}(\rho + 3p) &= -2\frac{\ddot{a}}{a} && (3) \end{aligned}$$

We get (3) by adding (1) and (2). The parameter k represents the tri-curvature and has values $-1, 0, 1$. The scale parameter is a and the dot represents differentiation with respect to time. The Hubble parameter is $H = \dot{a}/a$.

The solution for $k = 0$ and $p = 0$ is $a = a_0 t^{2/3}$. Then the critical density separating positive from negative curvature is $\rho_c = \frac{1}{6\pi G t^2}$. We also have $\frac{1}{a^3} = \frac{1}{a_0^3 t^2}$. Therefore $\rho_c = \frac{a_0^3}{6\pi G a^3}$ and therefore $\rho_c \propto a^{-3}$.

However, this conclusion only holds for $p = 0$. How does ρ_c depend on p? The rate of change of energy density is given by $\dot{\rho} = -3(\rho + p)H$. Dividing through by ρ, $\frac{\dot{\rho}}{\rho} = -3(1+p/\rho)H = -3(1+w)H$. From (1), the critical density $\rho_c = \frac{3H_c^2}{8\pi G}$ where H_c is the critical value of the Hubble parameter.

The rate of change

$$\dot{\rho}_c = \frac{6H_c\dot{H}_c}{8\pi G}$$

and therefore

$$\frac{\dot{\rho}_c}{\rho_c} = \frac{6H_c\dot{H}_c}{\rho_c 8\pi G} = -3(1+w)H_c$$

and consequently

$$\rho_c = -(\frac{\dot{H}_c}{4\pi G} + p)$$

We can see that $\dot{H}_c = \frac{a\ddot{a}-\dot{a}^2}{a^2} < 0$ must be exactly balanced with p to maintain critical density.

We can observe that $\rho > \rho_c$ requires $k = 1$ and $\rho < \rho_c$ requires $k = -1$. It has been estimated that the observable Universe contains about 4% of the matter required to achieve ρ_c which would seem to imply that $k = -1$, the Universe has negative curvature, and is expanding somewhat faster than the speed of light.

For a variety of reasons, this conclusion was unacceptable and it appeared necessary to fill the 96% with something. In the mid to late 1990s, observations were made which seemed to imply the expansion of the Universe was accelerating. Efforts were then made to revive the concept of a cosmological constant that would act in a repelling manner and would only be observable when the distances were quite large.

The cosmological equations were then modified to include the cosmological constant:

$$\frac{8\pi G\rho}{3} + \frac{\Lambda}{3} = \frac{k}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 \tag{4}$$

$$8\pi Gp - \Lambda = -2\frac{\ddot{a}}{a} - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 \tag{5}$$

Λ acts like a 'perfect fluid' with $\rho_\Lambda = \Lambda/8\pi G$ and $p_\Lambda = -\rho_\Lambda$.

So, we can substitute $\Lambda = 8\pi G\rho_\Lambda$ in (4) and $\Lambda = -8\pi Gp_\Lambda$ in (5) and get the equations:

$$\frac{8\pi G(\rho + \rho_\Lambda)}{3} = \frac{k}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 \tag{6}$$

$$8\pi Gp_\Lambda = -2\frac{\ddot{a}}{a} - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 \tag{7}$$

$$\frac{8\pi G}{3}(\rho + \rho_\Lambda + 3p_\Lambda) = -2\frac{\ddot{a}}{a} \tag{8}$$

We have set $p = 0$ in accordance with the Friedmann 'dust solution' where non-relativistic matter has negligible pressure. We also allow that ρ includes cold dark matter which is currently the fashion.

From (8) we can see that the limiting value (as $\rho \to 0$) of $\frac{\ddot{a}}{a} = \frac{\Lambda}{3}$. Using (6) and $k = 0$, the limiting value for the critical density is $\rho_\Lambda = constant$.

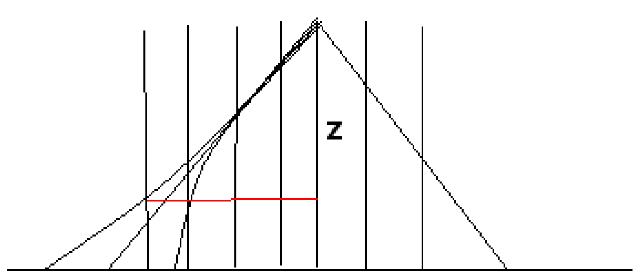
A *DeSitter Universe* is one with no matter but with a cosmological constant Λ. It describes a spatially flat Universe devoid of matter expanding exponentially with $a = e^{Ht}$ where H is the Hubble parameter $\dot{a}/a$. The λ-CDM model described above is asymptotically DeSitter with $H = \sqrt{\frac{\Lambda}{3}}$.

There is another empty solution that is seldom, if ever, discussed. That is the spatially negative solution. Letting $\rho = 0$ and $k = -1$ in equation (1), we get $\dot{a} = 1$. That is, the expansion rate is constant ($H = 1/a$) at the speed of light.

In accord with Occam's Razor we should ask whether Λ is necessary. Is an alternate view possible?

We could accept that the actual density of the Universe is 4% of critical and accept that it is not spatially flat. If $\rho < \rho_c$ then the Universe is dominated by negative curvature and in those regions that have negative curvature the expansion $\dot{a} \downarrow 1$ as $\rho \downarrow 0$.

If $r = l/a$ is a relatively small ($z < 1$) co-moving distance in flat space then $sinh(r)$ and $sin(r)$ are the corresponding co-moving distances in negatively and positively curved space respectively. Now, consider the following diagram:



The vertical lines represent the world-lines of galaxies given in co-moving coordinates. The central vertical line represents our galaxy. The bottom horizontal line represents the red-shift value $z = 1$. The sides of the isosceles triangle represent our past light cone under the assumption the Universe is spatially flat. The concave and convex lines represent our past light cone under the assumption of negative and positive spatial curvature respectively. The red line is the co-moving distance under the three different spatial assumptions. Under the assumption of flatness the distance is given by the well known linear Hubble relation $HD = cz$. For a given z, D will be slightly greater in the negative case and slightly less in the positive case than that predicted for a flat Universe. It has been observed that D is somewhat greater than the value predicted by the Hubble relation for a flat Universe.

We can represent the three optical distances* as follows (Carroll, Sean, 2019, p.348) and (Peebles, PJE, 1993, p.319):

For $k = +1$:

$$D_A = (1+z)^{-1} H_0^{-1} |\Omega_{k=+1}|^{-1/2} sin\left[|\Omega_{k=+1}|^{1/2} \int_0^z \frac{dz'}{E(z')}\right]$$

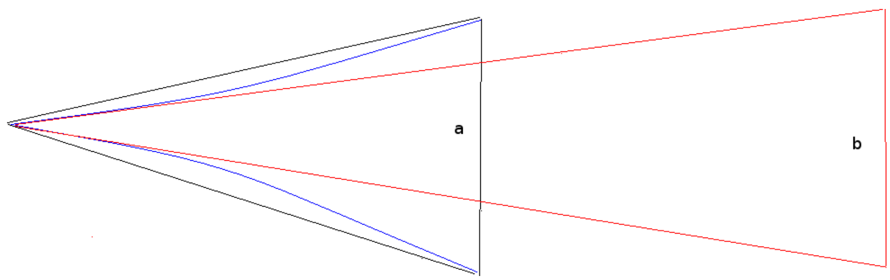
For $k = 0$:

$$D_A = (1+z)^{-1} H_0^{-1} \int_0^z \frac{dz'}{E(z')}$$

For $k = -1$:

$$D_A = (1+z)^{-1} H_0^{-1} |\Omega_{k=-1}|^{-1/2} sinh\left[|\Omega_{k=-1}|^{1/2} \int_0^z \frac{dz'}{E(z')}\right]$$

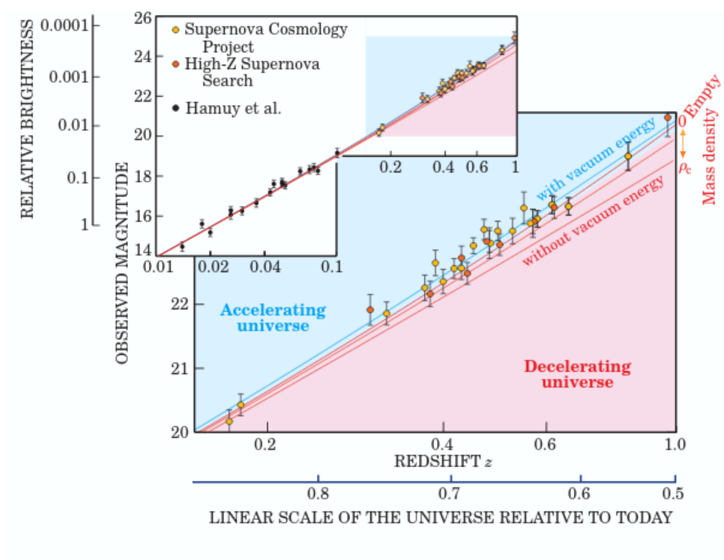
Consider two points of equal co-moving distance with one on the concave curve and the other on the straight line. As light passes through negatively curved space, parallel rays will diverge. So under the assumption of the inverse-square law, the optical distance using a standard candle (e.g. a super nova of known brightness) will appear greater (for the same co-moving distance) than if space were flat. Consequently, in a negatively curved space, the co-moving distance of galaxies is greater (for the same look-back time) than in flat space <u>and</u> they also appear even more distant due to the reduction in electromagnetic flux.



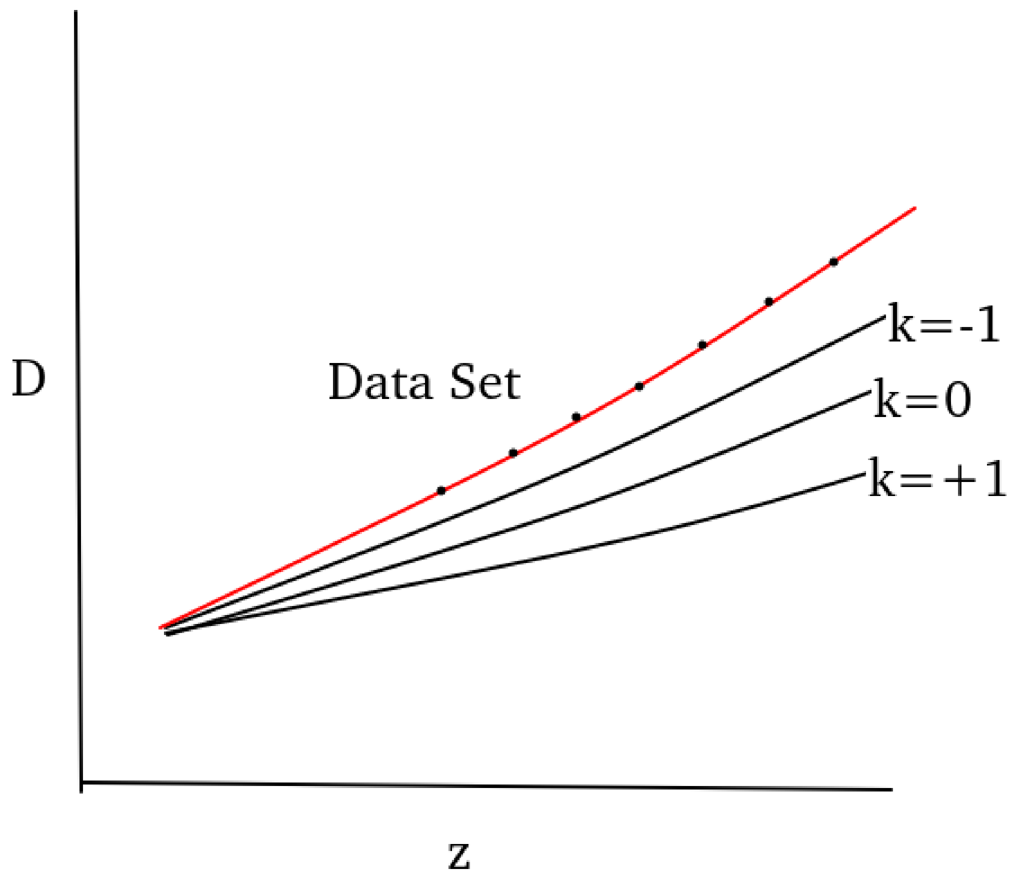
In the above figure the black lines represent rays traveling through flat space from **a** and converging on a focal point. The blue lines represent rays traveling from **a** through negatively curved space. The red triangle represents how the object appears further and dimmer at **b**.

In 2003, Saul Perlmutter discussed his supernova results for which he had won the Nobel proze in the previous decade. The reader may find this paper at http://www-supernova.lbl.gov/PhysicsTodayArticle.pdf

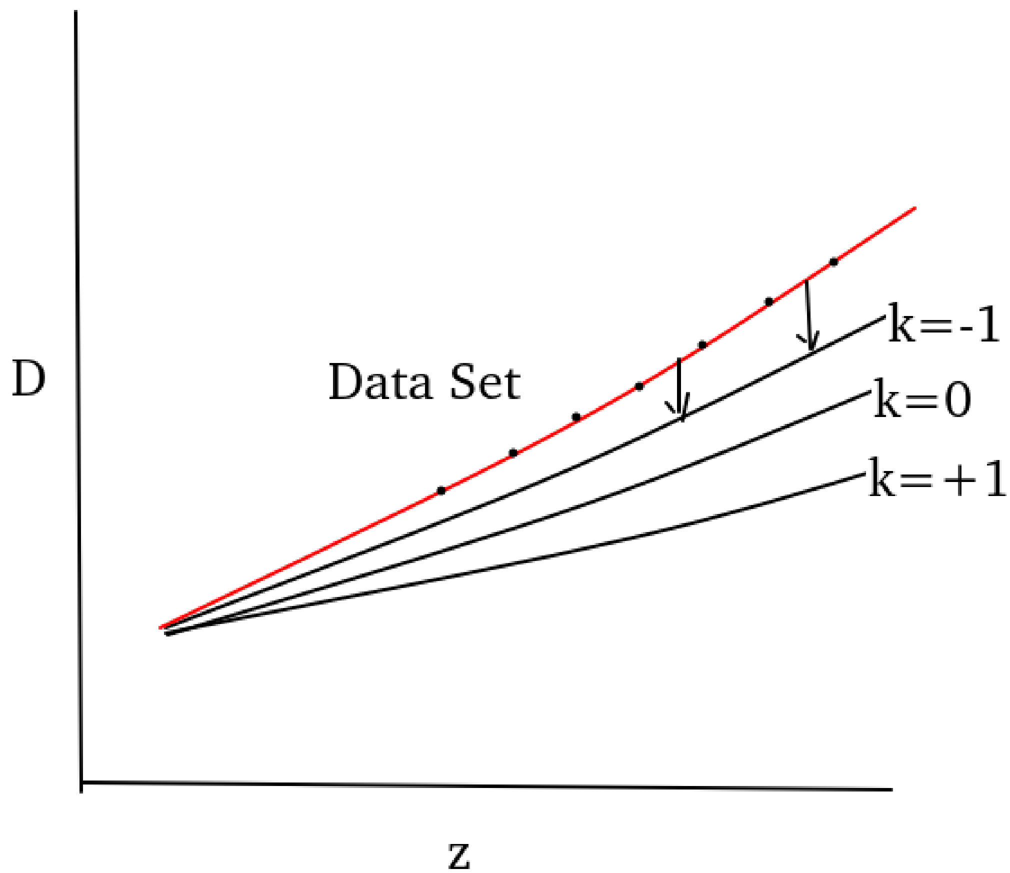
Essentially he had shown that the expansion of the Universe was speeding up and not slowing down as predicted by all models up to then. The following is a graphical excerpt from that paper.



And the following represents the data schematically:



The distance measurements were based on standard candle measurements of distances to supernovae with a known brightness. Here's the problem: If the inverse-square law was used to compute distances based on apparent brightness vs known brightness then possible errors can creep in. If the Universe were known to be flat the inverse-square law would work fine. However, curved space affects the apparent distance as shown three diagrams above. To be more precise, negative curvature would produce exactly what Perlmutter's data showed. The next diagram shows a correction for this effect bringing the data in line with what would be predicted by negative curvature.



We can actually use Perlmutter's data to determine the extent to which negative curvature affects apparent distance.

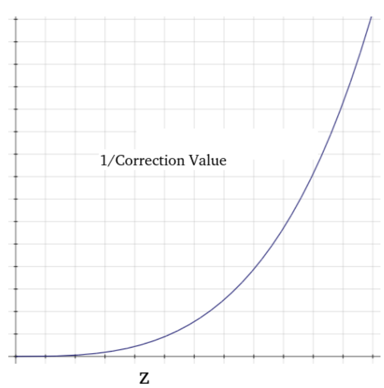
Let's consider this further. Under the consensus model ($\lambda - CDM$), $\rho + \rho_\Lambda = \rho_c$, where ρ_c is the critical density required for a flat Universe. ρ contains all sources of mass-energy excluding dark energy but including cold dark matter.

Using the third cosmological equation, $\frac{8\pi G}{3}(\rho + 3p) = -2\frac{\ddot{a}}{a}$ and setting $p = -\rho_\Lambda$ we get $\frac{8\pi G}{3}(\rho - 3\rho_\Lambda) = -2\frac{\ddot{a}}{a}$. In the past when $\rho - 3\rho_\Lambda > 0$ the expansion was slowing down and eventually since $\rho - 3\rho_\Lambda < 0$ the expansion has been accelerating. This is under the assumption the Universe is spatially flat.

Consider the alternate view that ρ consists of the known mass-energy of the Universe and is at present around 4% of ρ_c. Using the third cosmological equation again with $p = 0$, $\frac{8\pi G}{3}(\rho) = -2\frac{\ddot{a}}{a}$. So, $\ddot{a} < 0$ and is approaching zero as ρ decreases. There is a big difference between this and $\lambda - CDM$. We assert the difference is due to the errors introduced by using the inverse-square law for distances which relies on flat space. The differences arise when actually the Universe is spatially curved negatively as explained above. So, we may consider what we may call the Perlmutter et al procedure as a classic case of comparing apples to oranges.

First they calculate distances relative to red shift for the three curvatures using the D_A formulae. Then they calculate distance relative to redshift using the inverse-square law. But in negatively curved space things look further away than they really are, hence the deviation from all three models. Nevertheless, the $k = -1$ case is closest to the data. The interpretation closest to the facts would be that negatively curved space distorts distance measurements more than expected (based on the D_A formula) and there is a disagreement between distance measurements relative to red shift and those from using the inverse-square law (the latter relying on the flatness of space).

The exaggeration in distance measurement due to the negative curvature of space must be exponential. (Consider the case of concave mirrors facing each other.) So, an exponential correction value must be found to compensate.



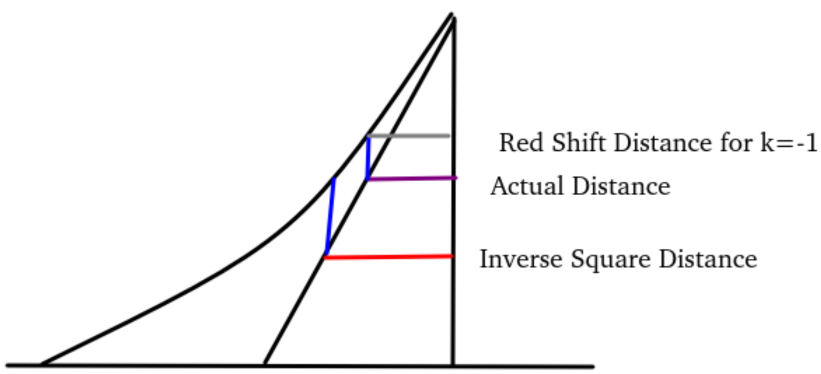
Using the R-W metric

$$
\begin{aligned}
d\sigma^2 &= a^2 sinh^2\chi d\Omega^2 && (9)\\
d\sigma^2 &= a^2 r^2 d\Omega^2 && (10)\\
d\sigma^2 &= a^2 sin^2\theta d\Omega^2 && (11)
\end{aligned}
$$

$d\Omega^2/d\sigma^2 = \frac{ApparentBrightness}{IntrinsicBrightness} = \frac{1}{(OpticalDistance)^2}$

Equation (10) is the inverse square law. Equation (9) shows how negative curvature affects the above ratio.

Using (10), $\frac{1}{D^2} = \frac{1}{a^2r^2}$. Using (9), $\frac{1}{D'^2} = \frac{1}{a^2 sinh^2\chi}$. Then $D' = \frac{sinh\chi}{r}D$. So the correction factor must be $\frac{r}{sinh\chi}$.



So, we have the distance based on the inverse square law and standard candles, the actual distance after applying the correction factor, and the distance based on red shift (D_A) for $k = -1$.

Our conclusion is that the cosmological constant Λ is an illusion resulting from the difference in distance calculation in flat space versus curved space.

*There are (at least) three types of distance to consider in cosmology:

1) The current distance which we call D_{now} which we cannot directly observe; 2) The lookback distance which is the lookback time multiplied by

the speed of light; 3) The apparent distance of an object from the point of view of an observer. This involves either calculating distance using red shift or using the apparent brightness of known standard candles versus their intrinsic brightness. It is this third group we refer to as *optical distance.*

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