Electromagnetism

Summary: Here we derive the Maxwell equations using the method of differential forms. We will define a 1-form A (a gauge) which we will show emerges as a relational decomposition of zero which we will denote by $0 \to \omega^2 + (*\omega)^2$. We will use this idea throughout the remainder of our articles, especially in *A Theory of Origins*. A relational decomposition of zero is, as the name implies, *relational* and is not dependent ontologically on anything else (except space-time) for its existence. We show later that space-time itself is also a relational decomposition of zero.

Let $A = A_t dt + A_x dx + A_y dy + A_z dz$ be the electromagnetic 1-form.

We calculate the Faraday 2-form as:

$$dA = (\partial_t A_t dt + \partial_x A_t dx + \partial_y A_t dy + \partial_z A_t dz) \wedge dt$$
$$+(\partial_t A_x dt + \partial_x A_x dx + \partial_y A_x dy + \partial_z A_x dz) \wedge dx$$
$$+(\partial_t A_y dt + \partial_x A_y dx + \partial_y A_y dy + \partial_z A_y dz) \wedge dy$$
$$+(\partial_t A_z dt + \partial_x A_z dx + \partial_y A_z dy + \partial_z A_z dz) \wedge dz$$

$$= (\partial_x A_t - \partial_t A_x) dx \wedge dt$$
$$+(\partial_y A_t - \partial_t A_y) dy \wedge dt$$
$$+(\partial_z A_t - \partial_t A_z) dz \wedge dt$$
$$+(\partial_x A_y - \partial_y A_x) dx \wedge dy$$
$$+(\partial_y A_z - \partial_z A_y) dy \wedge dz$$
$$+(\partial_z A_x - \partial_x A_z) dz \wedge dx$$

which is a vector in a 6 dimensional space of 2-forms. The dimension of the space of p-forms of a n-dimensional space, denoted $dim(\Lambda_n^p) = \begin{pmatrix} n \\ p \end{pmatrix}$ which in this case =6. The anticommutivity used above is based on a fact about forms that $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ where α is a k-form and β is an l-form. Also used above is the fact that for a 1-form α, $\alpha \wedge \alpha = -\alpha \wedge \alpha = 0$. Furthermore, notice that dA has the same form as the curl of a vector field but in a higher dimension where the curl is not defined. In the following we shall write $A_{x'}$, $B_{x'}$... as A'_x, B'_x... . Though the former is more technically correct, the latter is easier to read.

For simplicity we set:

$$B_x = \partial_y A_z - \partial_z A_y$$
$$B_y = \partial_z A_x - \partial_x A_z$$
$$B_z = \partial_x A_y - \partial_y A_x$$

which are the components of the magnetic field vector.

And

$$E_x = \partial_x A_t - \partial_t A_x$$
$$E_y = \partial_y A_t - \partial_t A_y$$
$$E_z = \partial_z A_t - \partial_t A_z$$

which are the components of the electric field vector.

We then write dA in terms of these components:

$$dA = E_x dx \wedge dt$$
$$+E_y dy \wedge dt$$
$$+E_z dz \wedge dt$$
$$+B_z dx \wedge dy$$
$$+B_x dy \wedge dz$$
$$+B_y dz \wedge dx$$

We next calculate $d(dA)$. We use the same method as for A.

$$d(dA) = (\partial_t E_x dt + \partial_x E_x dx + \partial_y E_x dy + \partial_z E_x dz) dx \wedge dt$$
$$+(\partial_t E_y dt + \partial_x E_y dx + \partial_y E_y dy + \partial_z E_y dz) dy \wedge dt$$
$$+(\partial_t E_z dt + \partial_x E_z dx + \partial_y E_z dy + \partial_z E_z dz) dz \wedge dt$$
$$+(\partial_t B_z dt + \partial_x B_z dx + \partial_y B_z dy + \partial_z B_z dz) dx \wedge dy$$
$$+(\partial_t B_x dt + \partial_x B_x dx + \partial_y B_x dy + \partial_z B_x dz) dy \wedge dz$$

$$+(\partial_t B_y dt + \partial_x B_y dx + \partial_y B_y dy + \partial_z B_y dz) dz \wedge dx$$

$$d(dA) = (\partial_x E_y - \partial_y E_x + \partial_t B_z) dx \wedge dy \wedge dt$$
$$+(\partial_y E_z - \partial_z E_y + \partial_t B_x) dy \wedge dz \wedge dt$$
$$+(\partial_z E_x - \partial_x E_z + \partial_t B_y) dz \wedge dx \wedge dt$$
$$+(\partial_x B_x + \partial_y B_y + \partial_z B_z) dx \wedge dy \wedge dz$$

which is a vector in a 4 dimensional space of 3-forms. That is, $dim(\Lambda_4^3) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4$. The anticommutivity used here is based on, for example,

$$dx \wedge dy \wedge dz = (dx \wedge dy) \wedge dz = (-dy \wedge dx) \wedge dz = -dy \wedge dx \wedge dz$$

It is a well established theorem in the calculus of forms that $d^2 = 0$. So, setting $d(dA) = 0$ we get from the first three equations $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and from the last $\nabla \cdot \mathbf{B} = 0$. These are two of Maxwell's equations in empty space.

To derive the other two equations we use the Maxwell 2-form which is the dual to the Faraday 2-form.

Expressing the dual by its components we have:

$$* \begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix}$$

Then

$$*dA = -B_x dx \wedge dt$$
$$-B_y dy \wedge dt$$
$$-B_z dz \wedge dt$$
$$+E_z dx \wedge dy$$
$$+E_x dy \wedge dz$$

$$+E_y dz \wedge dx$$

Then

$$d(*dA) = -(\partial_t B_x dt + \partial_x B_x dx + \partial_y B_x dy + \partial_z B_x dz) dx \wedge dt$$

$$-(\partial_t B_y dt + \partial_x B_y dx + \partial_y B_y dy + \partial_z B_y dz) dy \wedge dt$$

$$-(\partial_t B_z dt + \partial_x B_z dx + \partial_y B_z dy + \partial_z B_z dz) dz \wedge dt$$

$$+(\partial_t E_z dt + \partial_x E_z dx + \partial_y E_z dy + \partial_z E_z dz) dx \wedge dy$$

$$+(\partial_t E_x dt + \partial_x E_x dx + \partial_y E_x dy + \partial_z E_x dz) dy \wedge dz$$

$$+(\partial_t E_y dt + \partial_x E_y dx + \partial_y E_y dy + \partial_z E_y dz) dz \wedge dx$$

$$d(*dA) = (-\partial_x B_y + \partial_y B_x + \partial_t E_z) dx \wedge dy \wedge dt$$

$$+(-\partial_y B_z + \partial_z B_y + \partial_t E_x) dy \wedge dz \wedge dt$$

$$+(\partial_x B_z - \partial_z B_x + \partial_t E_y) dz \wedge dx \wedge dt$$

$$+(\partial_x E_x + \partial_y E_y + \partial_z E_z) dx \wedge dy \wedge dz$$

So, setting $d(*dA) = 0$ we get from the first three equations $\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$ and from the last $\nabla \cdot \mathbf{E} = 0$. These are the last two of Maxwell's equations in empty space. $d^2(A) = 0$ and $d * d(A) = 0$ are Lorentz scalars and hence invariant under a Lorentz transformation.

Lorentz covariance:

The Lorentz transformation describes space-time transformation between inertial frames in motion relative to each other. We assume for simplicity that the direction of motion is along the x-axis for each. Then the coordinates transform according to the Lorentz transformation:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

where $cosh(\alpha) = \frac{1}{\sqrt{1-v^2}}$ and $sinh(\alpha) = \frac{v}{\sqrt{1-v^2}}$. c is assumed to be =1. $cosh(\alpha)$ is always positive but $sinh(\alpha)$ can be positive or negative depending on the direction of motion.

It is clear that the 1-forms also transform as

$$\begin{pmatrix} dt' \\ dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

We begin first with the electromagnetic 1-form

$A = A_t dt + A_x dx + A_y dy + A_z dz$

In the alternate coordinates it is

$A' = A'_t dt' + A'_x dx' + A'_y dy' + A'_z dz'$

So,

$$A' = A'_t dt' + A'_x dx' + A'_y dy' + A'_z dz'$$
$$A' = A'_t (cosh(\alpha) dt + sinh(\alpha) dx)$$
$$+A'_x (sinh(\alpha) dt + (cosh(\alpha) dx)$$
$$+A'_y dy' + A'_z dz'$$

Grouping terms we get:

$$A' = (A'_t cosh(\alpha) + A'_x sinh(\alpha))dt$$
$$+(A'_t sinh(\alpha) + A'_x (cosh(\alpha))dx$$
$$+A'_y dy' + A'_z dz'$$

So, $A' = A$ if

$$\begin{pmatrix} A_t \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A'_t \\ A'_x \\ A'_y \\ A'_z \end{pmatrix}$$

More generally, let Λ be a coordinate transformation with a transpose Λ^T.

Let $A = \begin{pmatrix} A_t & A_x & A_y & A_z \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix} = [A]^T d\overline{x}$

and $A' = [A']^T d\overline{x}' = [A']^T \Lambda^T d\overline{x}$. Then $A = A'$ if $[A]^T = [A']^T \Lambda^T$

That is, $\begin{pmatrix} A_t \\ A_x \\ A_y \\ A_z \end{pmatrix} = \Lambda \begin{pmatrix} A'_t \\ A'_x \\ A'_y \\ A'_z \end{pmatrix}$

In a traditional development of electromagnetism, the field is given by

$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$

$\mathbf{B} = \nabla \times \mathbf{A}$

where the pair $(\phi, \mathbf{A})$ is called a *gauge*. The correspondence with our development here is $(A_t, A_x, A_y, A_z) = (\phi, A_x, A_y, A_z)$. That is, $\mathbf{A} = (A_x, A_y, A_z)$.

The traditional derivation of the Maxwell equations proceeds as follows:

Using the tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

and

$\overline{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$

$\overline{F}^{12} = \frac{1}{2}(F^{34} - F^{43}) = B_x$

$\overline{F}^{13} = \frac{1}{2}(-F^{24} + F^{42}) = -B_y$

$\overline{F}^{14} = \frac{1}{2}(F^{23} - F^{32}) = -B_z$

$\overline{F}^{21} = \frac{1}{2}(-F^{34} + F^{43}) = B_x$

$\overline{F}^{23} = \frac{1}{2}(F^{14} - F^{41}) = -B_z$

$\overline{F}^{24} = \frac{1}{2}(-F^{13} + F^{31}) = E_y$

$\overline{F}^{31} = \frac{1}{2}(F^{24} - F^{42}) = B_y$

$\overline{F}^{32} = \frac{1}{2}(-F^{14} + F^{41}) = E_z$

$\overline{F}^{34} = \frac{1}{2}(F^{12} - F^{21}) = -E_x$

$\overline{F}^{41} = \frac{1}{2}(-F^{23} + F^{32}) = B_z$

$\overline{F}^{42} = \frac{1}{2}(F^{13} - F^{31}) = -E_y$

$\overline{F}^{43} = \frac{1}{2}(-F^{12} + F^{21}) = E_x$

$$\overline{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$\begin{aligned}
\partial_\nu \overline{F}^{1\nu} &= -\partial_x B_x - \partial_y B_y - \partial_z B_z \\
\partial_\nu \overline{F}^{2\nu} &= \partial_t B_x + \partial_z E_y - \partial_y E_z \\
\partial_\nu \overline{F}^{3\nu} &= \partial_t B_y - \partial_x E_z + \partial_z E_x \\
\partial_\nu \overline{F}^{4\nu} &= \partial_t B_z + \partial_x E_y - \partial_y E_x
\end{aligned}$$

Then $\begin{pmatrix} \partial_\nu \overline{F}^{1\nu} \\ \partial_\nu \overline{F}^{2\nu} \\ \partial_\nu \overline{F}^{3\nu} \\ \partial_\nu \overline{F}^{4\nu} \end{pmatrix} = \mathbf{0}$

implies

$$\begin{pmatrix} -\partial_x B_x - \partial_y B_y - \partial_z B_z \\ \partial_t B_x + \partial_z E_y - \partial_y E_z \\ \partial_t B_y - \partial_x E_z + \partial_z E_x \\ \partial_t B_z + \partial_x E_y - \partial_y E_x \end{pmatrix} = \begin{pmatrix} -\nabla \cdot \mathbf{B} \\ (\nabla \times \mathbf{E} + \partial_t \mathbf{B})_x \\ (\nabla \times \mathbf{E} + \partial_t \mathbf{B})_y \\ (\nabla \times \mathbf{E} + \partial_t \mathbf{B})_z \end{pmatrix} = \mathbf{0}$$

Then $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\begin{aligned} \partial_\nu F^{1\nu} &= -\partial_x E_x - \partial_y E_y - \partial_z E_z \\ \partial_\nu F^{2\nu} &= \partial_t E_x - \partial_y B_z + \partial_z B_y \\ \partial_\nu F^{3\nu} &= \partial_t E_y + \partial_x B_z - \partial_z B_x \\ \partial_\nu F^{4\nu} &= \partial_t E_z - \partial_x B_y + \partial_y B_x \end{aligned}$$

Then $\begin{pmatrix} \partial_\nu F^{1\nu} \\ \partial_\nu F^{2\nu} \\ \partial_\nu F^{3\nu} \\ \partial_\nu F^{4\nu} \end{pmatrix} = \mathbf{0}$

implies

$$\begin{pmatrix} -\partial_x E_x - \partial_y E_y - \partial_z E_z \\ \partial_t E_x - \partial_y B_z + \partial_z B_y \\ \partial_t E_y + \partial_x B_z - \partial_z B_x \\ \partial_t E_z - \partial_x B_y + \partial_y B_x \end{pmatrix} = \begin{pmatrix} -\nabla \cdot \mathbf{E} \\ (\partial_t \mathbf{E} - \nabla \times \mathbf{B})_x \\ (\partial_t \mathbf{E} - \nabla \times \mathbf{B})_y \\ (\partial_t \mathbf{E} - \nabla \times \mathbf{B})_z \end{pmatrix} = \mathbf{0}$$

Then $\nabla \cdot \mathbf{E} = 0$ and $-\nabla \times \mathbf{B} + \partial_t \mathbf{E} = \mathbf{0}$

So, in flat space-time $\partial_\nu F^{\mu\nu} = \partial_\nu \overline{F}^{\mu\nu} = 0$ is equivalent to the Maxwell equations. In general, $\nabla_\nu F^{\mu\nu} = \nabla_\nu \overline{F}^{\mu\nu} = 0$ where ∇_σ is the covariant derivative.

The Lorentz co-variance of the 1-form A corresponds to the Lorentz co-variance of the 4-vector (ϕ, A_x, A_y, A_z). That is, $\phi^2 - A_x^2 - A_y^2 - A_z^2$ is invariant under Lorentz transformations.

Let Λ be a general Lorentz transformation. It is known that Λ preserves the Minkowski metric $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

where $||\mathbf{V}||^2 = \begin{pmatrix} V_0 & V_1 & V_2 & V_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$

That is, $\Lambda^T g \Lambda = g$

Setting $[A]^T = \begin{pmatrix} A_t & A_x & A_y & A_z \end{pmatrix}$ we then have:

$A_t^2 - A_x^2 - A_y^2 - A_z^2 = [A]^T g[A] = (\Lambda[A'])^T g \Lambda[A'] = [A']^T \Lambda^T g \Lambda[A'] = [A']^T g[A']$.

Thus the co-variance of the 1-form establishes the co-variance of the gauge.

Now, for the Faraday 2-form (with the boost direction along the x-axis):

$$\begin{gathered} dA' = E'_x dx' \wedge dt' \\ +E'_y dy' \wedge dt' \\ +E'_z dz' \wedge dt' \\ +B'_z dx' \wedge dy' \\ +B'_x dy' \wedge dz' \\ +B'_y dz' \wedge dx' \end{gathered}$$

Then

$$dA' = E'_x(sinh(\alpha)dt + cosh(\alpha)dx) \wedge (cosh(\alpha)dt + sinh(\alpha)dx)$$
$$+E'_y dy' \wedge (cosh(\alpha)dt + sinh(\alpha)dx)$$
$$+E'_z dz' \wedge (cosh(\alpha)dt + sinh(\alpha)dx)$$
$$+B'_z(sinh(\alpha)dt + cosh(\alpha)dx) \wedge dy'$$
$$+B'_x dy' \wedge dz'$$
$$+B'_y dz' \wedge (sinh(\alpha)dt + cosh(\alpha)dx)$$

and grouping terms gives:

$$dA' = E'_x dx \wedge dt$$
$$+(E'_y cosh(\alpha) - B'_z sinh(\alpha))dy \wedge dt$$
$$+(E'_z cosh(\alpha) + B'_y sinh(\alpha))dz \wedge dt$$
$$+(B'_z cosh(\alpha) - E'_y sinh(\alpha))dx \wedge dy$$
$$+B'_x dy \wedge dz$$
$$+(B'_y cosh(\alpha) + E'_z sinh(\alpha))dz \wedge dx$$

So, $dA' = dA$ if

$$\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & cosh(\alpha) & 0 & 0 & 0 & -sinh(\alpha) \\ 0 & 0 & cosh(\alpha) & 0 & sinh(\alpha) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & sinh(\alpha) & 0 & cosh(\alpha) & 0 \\ 0 & -sinh(\alpha) & 0 & 0 & 0 & cosh(\alpha) \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \\ E'_z \\ B'_x \\ B'_y \\ B'_z \end{pmatrix}$$

Let $\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & cosh(\alpha) & 0 & 0 & 0 & -sinh(\alpha) \\ 0 & 0 & cosh(\alpha) & 0 & sinh(\alpha) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & sinh(\alpha) & 0 & cosh(\alpha) & 0 \\ 0 & -sinh(\alpha) & 0 & 0 & 0 & cosh(\alpha) \end{pmatrix}$

and $M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

Then $M = \Phi^T M \Phi$

and consequently,

$E^2 - B^2$

$$= \begin{pmatrix} E_x & E_y & E_z & B_x & B_y & B_z \end{pmatrix} M \begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix}$$

$$= \begin{pmatrix} E'_x & E'_y & E'_z & B'_x & B'_y & B'_z \end{pmatrix} \Phi^T M \Phi \begin{pmatrix} E'_x \\ E'_y \\ E'_z \\ B'_x \\ B'_y \\ B'_z \end{pmatrix}$$

$$= \begin{pmatrix} E'_x & E'_y & E'_z & B'_x & B'_y & B'_z \end{pmatrix} M \begin{pmatrix} E'_x \\ E'_y \\ E'_z \\ B'_x \\ B'_y \\ B'_z \end{pmatrix}$$

$= (E')^2 - (B')^2$

We can make the more general claim that the 6 components of any 2-

form $dM = F_x dx \wedge dt + F_y dy \wedge dt + F_z dz \wedge dt + G_z dx \wedge dy + G_x dy \wedge dz + G_y dz \wedge dx$

and hence any 6-vector, will transform in the same way. That is,

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ G_x \\ G_y \\ G_z \end{pmatrix} = \Phi \begin{pmatrix} F'_x \\ F'_y \\ F'_z \\ G'_x \\ G'_y \\ G'_z \end{pmatrix}$$

Let $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ be quaternions with multiplication table:

$\times$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be Pauli spin matrices with multiplication table:

$\times$	I	σ_1	σ_2	σ_3
I	I	σ_1	σ_2	σ_3
σ_1	σ_1	I	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	I	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	I

There is an isomorphism from the span of the quaternions $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to the span of $\{I, i\sigma_1, i\sigma_2, i\sigma_3,\}$ given by $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (I, -i\sigma_1, -i\sigma_2, -i\sigma_3)$ or by $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (I, i\sigma_3, i\sigma_2, i\sigma_1)$.

Let $\mathbf{M} = \mathbf{1}M$

That is, $\mathbf{M} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{1} \end{pmatrix}$

We can identify $I \leftrightarrow \mathbf{1}$ through either of the above isomorphisms and get as a square root

$$\sqrt{\mathbf{M}} = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{j} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{k} \end{pmatrix}$$

Then $M = \Phi^T M \Phi$ implies $\mathbf{M} = (\sqrt{\mathbf{M}}\Phi)^T \sqrt{\mathbf{M}}\Phi$ and for the electromagnetic 6-vector $\mathbf{V}'$,

$$\mathbf{V}^T\mathbf{M}\mathbf{V} = (\sqrt{\mathbf{M}}\mathbf{V})^T\sqrt{\mathbf{M}}\mathbf{V} = (\sqrt{\mathbf{M}}\Phi\mathbf{V}')^T\sqrt{\mathbf{M}}\Phi\mathbf{V}' = (\mathbf{V}')^T\mathbf{M}\mathbf{V}'.$$

The dual $*dA$ should be invariant under the same transformation but for completeness we show this:

$$\begin{aligned} *dA' = &-B'_x dx' \wedge dt' \\ &-B'_y dy' \wedge dt' \\ &-B'_z dz' \wedge dt' \\ &+E'_z dx' \wedge dy' \\ &+E'_x dy' \wedge dz' \\ &+E'_y dz' \wedge dx' \end{aligned}$$

$$*dA' = -B'_x(sinh(\alpha)dt + (cosh(\alpha)dx) \wedge (cosh(\alpha)dt + sinh(\alpha)dx)$$

$$-B'_y dy' \wedge (cosh(\alpha)dt + sinh(\alpha)dx)$$

$$-B'_z dz' \wedge (cosh(\alpha)dt + sinh(\alpha)dx)$$
$$+E'_z(sinh(\alpha)dt + (cosh(\alpha)dx) \wedge dy'$$
$$+E'_x dy' \wedge dz'$$
$$+E'_y dz' \wedge (sinh(\alpha)dt + (cosh(\alpha)dx)$$

and then grouping terms we get:

$$*dA' = -B'_x dx \wedge dt$$
$$-(B'_y cosh(\alpha) + E'_z sinh(\alpha))dy \wedge dt$$
$$-(B'_z cosh(\alpha) - E'_y sinh(\alpha))dz \wedge dt$$
$$+(E'_z cosh(\alpha) + B'_y sinh(\alpha))dx \wedge dy$$
$$+E'_x dy \wedge dz$$
$$+(E'_y cosh(\alpha) - B'_z sinh(\alpha))dz \wedge dx$$

Then $*dA = *dA'$ if

$$\begin{pmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & cosh(\alpha) & 0 & 0 & 0 & -sinh(\alpha) \\ 0 & 0 & cosh(\alpha) & 0 & sinh(\alpha) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & sinh(\alpha) & 0 & cosh(\alpha) & 0 \\ 0 & -sinh(\alpha) & 0 & 0 & 0 & cosh(\alpha) \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \\ E'_z \\ B'_x \\ B'_y \\ B'_z \end{pmatrix}$$

which is the same transformation making dA invariant.

$d^2A = 0$ and $d * dA = 0$ are invariant (as scalars) under the above transformations.

Maxwell's four equations in vector notation are:

$$\nabla \cdot \mathbf{E} = 0; -\nabla \times \mathbf{B} + \partial_t \mathbf{E} = \mathbf{0}$$

$$\nabla \cdot \mathbf{B} = 0; \nabla \times \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}$$

The explicitly co-variant form of the above equations is:

$$\nabla \cdot \mathbf{E} = 0; -\nabla \times \mathbf{B} + \nabla_t \mathbf{E} = \mathbf{0}$$

$$\nabla \cdot \mathbf{B} = 0; \nabla \times \mathbf{E} + \nabla_t \mathbf{B} = \mathbf{0}$$

where ∇_t is the co-variant derivative.

As already observed, Lorentz transformations preserve the Minkowski metric. That is $g = \Lambda^T g \Lambda$. In Minkowski space $\partial_t \mathbf{E} = \nabla_t \mathbf{E}$ and $\partial_t \mathbf{B} = \nabla_t \mathbf{B}$.

Therefore, the Maxwell equations are co-variant under a Lorentz transformation Λ.

Example:

Suppose the boost direction is x and also suppose an electromagnetic wave originating at some $x > 0$ and measured in (t, x, y, z) as $\mathbf{E} = cos(kx - \omega t)\mathbf{e}_y$ and $\mathbf{B} = cos(kx - \omega t)\mathbf{e}_z$. Then the same wave measured in (t', x', y, z) would be $\mathbf{E}' = cos(k'x' - \omega' t')\mathbf{e}_y$ and $\mathbf{B}' = cos(k'x' - \omega' t')\mathbf{e}_z$.

Let $\eta^{-1} = \sqrt{\frac{1-v}{1+v}}$. Then the wavelength $\lambda' = \eta^{-1}\lambda$. So $k' = \frac{2\pi}{\lambda'} = \eta k$.

Then for the moving frame $\mathbf{E}' = cos(\eta k x' - \omega' t')\mathbf{e}_y$

and $\mathbf{B}' = cos(\eta k x' - \omega' t')\mathbf{e}_z$.

Then $\partial_{t'} \mathbf{E}' = \omega' sin(\eta k x' - \omega' t')\mathbf{e}_y$

and $(\nabla \times \mathbf{B}') = \eta k sin(\eta k x' - \omega' t')\mathbf{e}_y$.

Then $\partial_{t'} \mathbf{B}' = \omega' sin(\eta k x' - \omega' t')\mathbf{e}_z$

and $(\nabla \times \mathbf{E}') = -\eta k sin(\eta k x' - \omega' t')\mathbf{e}_z$.

In order to preserve the canonical form of the Maxwell equations we set

$$\partial_{t'}\mathbf{E}' = \omega' sin(\eta k x' - \omega' t')\mathbf{e}_y = \eta k sin(\eta k x' - \omega' t')\mathbf{e}_y$$

and

$$\partial_{t'}\mathbf{B}' = \omega' sin(\eta k x' - \omega' t')\mathbf{e}_z = \eta k sin(\eta k x' - \omega' t')\mathbf{e}_z$$

This implies $\omega' = \eta k$. So, the wave in the moving frame is

$\mathbf{E}' = cos(\sqrt{\frac{1+v}{1-v}}kx' - \sqrt{\frac{1+v}{1-v}}kt')\mathbf{e}_y$ and $\mathbf{B}' = cos(\sqrt{\frac{1+v}{1-v}}kx' - \sqrt{\frac{1+v}{1-v}}kt')\mathbf{e}_z$.

We now derive the following formulas:

$$dA^2 = dA \wedge dA = 2(\mathbf{B} \cdot \mathbf{E})dx \wedge dy \wedge dz \wedge dt$$

and

$$(*dA)^2 = *dA \wedge *dA = -2(\mathbf{B} \cdot \mathbf{E})dx \wedge dy \wedge dz \wedge dt$$

and conclude that $(dA)^2 + (*dA)^2 = 0$, which expresses the fundamental fact of electromagnetic duality. We also conclude that $*^2 = -1$ which is implied by

$$* = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Derivation:

$$\begin{gathered} dA = E_x dx \wedge dt \\ +E_y dy \wedge dt \\ +E_z dz \wedge dt \\ +B_z dx \wedge dy \\ +B_x dy \wedge dz \\ +B_y dz \wedge dx \end{gathered}$$

Then

$$dA^2 = dA \wedge dA$$
$$= B_x E_x dy \wedge dz \wedge dx \wedge dt + E_x B_x dx \wedge dt \wedge dy \wedge dz$$
$$+ B_y E_y dz \wedge dx \wedge dy \wedge dt + E_y B_y dy \wedge dt \wedge dz \wedge dx$$
$$+ B_z E_z dx \wedge dy \wedge dz \wedge dt + E_z B_z dz \wedge dt \wedge dx \wedge dy$$
$$= (2B_x E_x + 2B_y E_y + 2B_z E_z) dx \wedge dy \wedge dz \wedge dt$$
$$= 2(\mathbf{B} \cdot \mathbf{E}) dx \wedge dy \wedge dz \wedge dt$$

$$*dA = -B_x dx \wedge dt$$
$$-B_y dy \wedge dt$$
$$-B_z dz \wedge dt$$
$$+E_z dx \wedge dy$$
$$+E_x dy \wedge dz$$
$$+E_y dz \wedge dx$$

$$(*dA)^2 = *dA \wedge *dA$$
$$= -B_x E_x dx \wedge dt \wedge dy \wedge dz - E_x B_x dy \wedge dz \wedge dx \wedge dt$$
$$-B_y E_y dy \wedge dt \wedge dz \wedge dx - E_y B_y dz \wedge dx \wedge dy \wedge dt$$
$$-B_z E_z dz \wedge dt \wedge dx \wedge dy - E_z B_z dx \wedge dy \wedge dz \wedge dt$$
$$= (-2B_x E_x - 2B_y E_y - 2B_z E_z) dx \wedge dy \wedge dz \wedge dt$$
$$= -2(\mathbf{B} \cdot \mathbf{E}) dx \wedge dy \wedge dz \wedge dt$$

We can then write $0 \to (dA)^2 + (*dA)^2$

The Constancy of the Speed of Light

Let $\Delta s^2 = \Delta t^2 - \Delta x^2$ and $\Delta s^2 = \Delta t'^2 - \Delta x'^2$ be space-time intervals for a light ray with respect to two inertial coordinate systems and

Let Λ be a Lorentz transform (e.g. $\Lambda = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) \\ sinh(\alpha) & cosh(\alpha) \end{pmatrix}$).

$\Delta s^2 = \begin{pmatrix} \Delta t' & \Delta x' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix}$

$\Delta s^2 = \begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix}$

$\Delta s^2 = \left[\Lambda \begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix} \right]^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix}$

$= \begin{pmatrix} \Delta t & \Delta x \end{pmatrix} \Lambda^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix}$

$= \begin{pmatrix} \Delta t & \Delta x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \end{pmatrix}$

So, if Δs^2 is zero in one case it equals zero in the other.

Mass is a Relativistic Invariant

The energy E and momentum P of a mass m traveling with uniform velocity v are given by

$E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ and $P = \frac{mv}{\sqrt{1-v^2/c^2}}$

$\frac{1}{1-v^2/c^2} - \frac{v^2/c^2}{1-v^2/c^2} = \frac{1-v^2/c^2}{1-v^2/c^2} = 1$

Then $\frac{m^2c^2}{1-v^2/c^2} - \frac{m^2v^2}{1-v^2/c^2} = m^2c^2$ and $(E/c)^2 - P^2 = m^2c^2$

Let Λ be a Lorentz transform (e.g. $\Lambda = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) \\ sinh(\alpha) & cosh(\alpha) \end{pmatrix}$).

$m^2c^2 = \begin{pmatrix} E'/c & P' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} E'/c \\ P' \end{pmatrix}$

$$m^2c^2 = \begin{pmatrix} E'/c \\ P' \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} E'/c \\ P' \end{pmatrix}$$

$$m^2c^2 = \left[\Lambda \begin{pmatrix} E/c \\ P \end{pmatrix}\right]^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} E/c \\ P \end{pmatrix}$$

$$= \begin{pmatrix} E/c & P \end{pmatrix} \Lambda^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} E/c \\ P \end{pmatrix}$$

$$= \begin{pmatrix} E/c & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} E/c \\ P \end{pmatrix}$$

Relativistic Kinetic energy

$K = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$ where m is the rest mass of a body and v is its uniform velocity. As a body approaches the speed of light, the kinetic energy approaches infinity but the mass remains the rest mass.

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