Electron Spin

It is well known that electrons (and other fermions) exhibit spin properties such as a magnetic dipole and spin angular momentum but the spin is not directly observed. In this article we attempt to demonstrate how this occurs.

The Pauli matrices are defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Their multiplication table is:

$\times$	I	σ_1	σ_2	σ_3
I	I	σ_1	σ_2	σ_3
σ_1	σ_1	I	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	I	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	I

There is an isomorphism from the span of the quaternions $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to the span of $\{I, i\sigma_1, i\sigma_2, i\sigma_3,\}$ given by $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (I, -i\sigma_1, -i\sigma_2, -i\sigma_3)$ or by $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (I, i\sigma_3, i\sigma_2, i\sigma_1)$ or $(\mathbf{1}, i\mathbf{i}, i\mathbf{j}, i\mathbf{k}) \leftrightarrow (I, \sigma_1, \sigma_2, \sigma_3)$.

Consider the following matrices in $M_{4\times 4}\{0, \pm 1\}^*$:

$$E_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$E_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$E_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$E_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Their multiplication table is

$\times$	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_0	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_1	E_1	$-E_0$	E_3	$-E_2$	E_5	$-E_4$	E_7	$-E_6$
E_2	E_2	E_3	$-E_0$	$-E_1$	E_6	E_7	$-E_4$	$-E_5$
E_3	E_3	$-E_2$	$-E_1$	E_0	E_7	$-E_6$	$-E_5$	E_4
E_4	E_4	E_5	$-E_6$	$-E_7$	$-E_0$	$-E_1$	E_2	E_3
E_5	E_5	$-E_4$	$-E_7$	E_6	$-E_1$	E_0	E_3	$-E_2$
E_6	E_6	E_7	E_4	E_5	$-E_2$	$-E_3$	$-E_0$	$-E_1$
E_7	E_7	$-E_6$	E_5	$-E_4$	$-E_3$	E_2	$-E_1$	E_0

We observe that

$\times$	E_0	E_2	E_4	E_6
E_0	E_0	E_2	E_4	E_6
E_2	E_2	$-E_0$	E_6	$-E_4$
E_4	E_4	$-E_6$	$-E_0$	E_2
E_6	E_6	E_4	$-E_2$	$-E_0$

has the same multiplicative structure as the quaternions:

$\times$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

giving the isomorphism $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (E_0, E_2, E_4, E_6)$.

Using the correspondence $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (I, -i\sigma_1, -i\sigma_2, -i\sigma_3)$,

$\times$	E_0	E_3	E_5	E_7
E_0	E_0	E_3	E_5	E_7
E_3	E_3	E_0	$-E_6$	E_4
E_5	E_5	E_6	E_0	$-E_2$
E_7	E_7	$-E_4$	E_2	E_0

has the same multiplicative structure as the Pauli matrices:

$\times$	I	σ_1	σ_2	σ_3
I	I	σ_1	σ_2	σ_3
σ_1	σ_1	I	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	I	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	I

giving the isomorphism $(I, \sigma_1, \sigma_2, \sigma_3) \leftrightarrow (E_0, E_3, E_5, E_7)$.

Let us propose that activities strictly within the span of the Pauli matrices are not observable except through interactions with the quaternion quadrant. We will return to why this might be so.

We propose that analogous to the cross product in classical physics we use the Lie bracket $\frac{1}{2}[A, B] = \frac{1}{2}(AB - BA)$. This describes rotation in the A, B plane whose axis is given by the bracket.

Consider a rotation in the E_3,E_5 plane. Considering that this rotation occurs inside the Pauli span it is unobservable. However,

$$\frac{1}{2}[E_3, E_5] = \frac{1}{2}(E_3E_5 - E_5E_3) = -E_6$$

, the latter product being in the quaternion realm and observable. So, the axis of rotation is observable with the spin angular momentum and magnetic dipole aligned with it.

We see the same thing is true for the other spin axes

$$\frac{1}{2}[E_3, E_7] = \frac{1}{2}(E_3E_7 - E_7E_3) = -E_4$$

and

$$\frac{1}{2}[E_5, E_7] = \frac{1}{2}(E_5E_7 - E_7E_5) = -E_2$$

.

Why might this be so?

Consider the span of the Pauli matrices

$$\mathbf{S} = \{\tau\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3 : \tau, x, y, z \in \mathbf{R}\}$$

Exponentiation gives us (using $(\mathbf{1}, i\mathbf{i}, i\mathbf{j}, i\mathbf{k}) \leftrightarrow (I, \sigma_1, \sigma_2, \sigma_3)$)

$$||exp\begin{pmatrix} 0 & -ix & iy & -iz \\ ix & 0 & iz & iy \\ -iy & -iz & 0 & ix \\ iz & -iy & -ix & 0 \end{pmatrix}||^2 = [\alpha\mathbf{1}+\beta(ix\mathbf{i}+iy\mathbf{j}+iz\mathbf{k})][\alpha\mathbf{1}-\beta(ix\mathbf{i}+iy\mathbf{j}+iz\mathbf{k})]$$

$= \alpha^2 - \beta^2x^2 - \beta^2y^2 - \beta^2z^2 = 1$ (a 3-PseudoSphere)

where $\alpha = \Sigma_{n=0}^{\infty}\frac{(x^2+y^2+z^2)^n}{(2n)!}$ and $\beta = \Sigma_{n=0}^{\infty}\frac{(x^2+y^2+z^2)^n}{(2n+1)!}$

(See the article **The Zero Point of Time**)

So, $span\{\sigma_1, \sigma_2, \sigma_3\}$ is tangent to our physical space and only interacts with it according to the above multiplicative structure of $\{E_0, E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$. But the Pauli matrices act as infinitesimal generators for the 3-PseudoSphere S_P^3 so we conclude $e^{\mathbf{S}} \sim \mathbf{R}^+ \times S_P^3$ is the geometry of our Universe.

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