Quaternion Space-time (Part 3):

Quaternion Electroweak Interactions

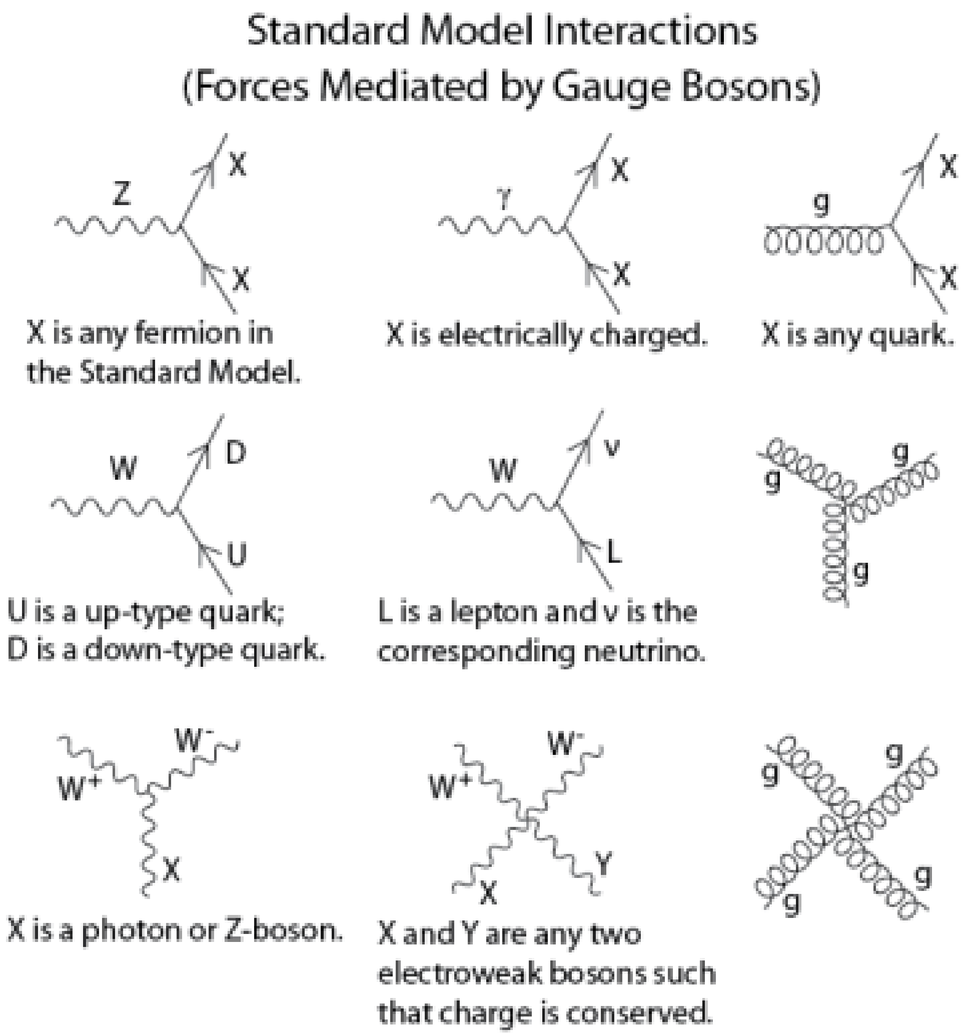

Image Source: $http://en.wikipedia.org/wiki/Standard\_Model\#mediaviewer/File:$
$Standard\_Model\_Feynman\_Diagram_Vertices.png$

We can express some of these interactions as:

(1) $W^+ \to \overline{D} + U$ where the overline represents an antiparticle.

(2) $W^- \to D + \overline{U}$

(3) $Z \to X + \overline{X}$

(4) $W^+ \to e^+ + \nu_e$

(5) $W^- \to e^- + \overline{\nu}_e$

From (2) and (5) we get neutron decay (i.e. beta decay):

From (2): $D \to U + W^-$ and from (5): $W^- \to e^- + \overline{\nu}_e$

Then $UDD \to UUD + e^- + \overline{\nu}_e$

That is, $n \to p^+ + e^- + \overline{\nu}_e$

The group which describes the symmetries of such interactions is $SU(2)$.

$SU(2) = \{\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1\}$

where the overline indicates complex conjugation.

$\{\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbf{C}\}$
$= \{\begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix} : a, b, c, d \in \mathbf{R}\}$
$= \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} ib & 0 \\ 0 & -ib \end{pmatrix} + \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & +id \\ id & 0 \end{pmatrix} : a, b, c, d \in \mathbf{R}\}$
$= \{a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
$: a, b, c, d \in \mathbf{R}\}$

Then $SU(2) = \{\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1\}$

$$= \{a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$: a, b, c, d \in \mathbf{R}, a^2 + b^2 + c^2 + d^2 = 1\}$$

And there is an isomorphism

$$(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}) \leftrightarrow (\mathbf{1}, \mathbf{i}, -\mathbf{j}, \mathbf{k})$$

So, we can also express $SU(2)$ in quaternion form as

$$SU(2) \simeq \{a\mathbf{1} + b\mathbf{i} - c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbf{R}, a^2 + b^2 + c^2 + d^2 = 1\}.$$

The set of generators of $SU(2)$ is $\{\mathbf{i}, -\mathbf{j}, \mathbf{k}\}$ which in the Standard Model is to be identified with the set of particles $\{W^+, W^-, Z\}$.

The multiplication table for this set of generators is

$\times$	$\mathbf{i}$	$-\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$-\mathbf{1}$	$-\mathbf{k}$	$-\mathbf{j}$
$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$

and that for the corresponding quaternions

$\times$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

So, any group which contains $\{\mathbf{i}, -\mathbf{j}, \mathbf{k}\}$ as a subset must also contain the quaternion group $\{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ as a subgroup.

The quaternions are associated with spatial rotations of the form

$\mathbf{p}' = R_{(\mathbf{q},\theta)}(\mathbf{p}) = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ where $\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is an initial vector before rotation, $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ is an axis of rotation with $||\mathbf{u}|| = 1$, θ is an angle of rotation, and

$\mathbf{q} = exp[\frac{\theta}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})] = cos\frac{\theta}{2}\mathbf{1} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})sin\frac{\theta}{2}$ and

$\mathbf{q}^{-1} = exp[-\frac{\theta}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})] = cos\frac{\theta}{2}\mathbf{1} - (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})sin\frac{\theta}{2}$

define the rotation.

The Lie group $SO(3)$ represents all spatial rotations in $\mathbf{R}^3$ so

$\{R_{(\mathbf{q},\theta)} : \mathbf{q} = exp[\frac{\theta}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})], 0 \leq \theta < 2\pi\} \simeq SO(3)$.

So, we need to clarify the relation between $SU(2)$ and $SO(3)$.

Recalling from above the isomorphism

$$(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}) \leftrightarrow (\mathbf{1}, \mathbf{i}, -\mathbf{j}, \mathbf{k})$$

which can also be written as

$$(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}) \leftrightarrow (\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$$

we can define the 2-to-1 surjective homomorphism $\phi : SU(2) \to SO(3)$

by $\phi(A) = -\frac{1}{2}\begin{pmatrix} Tr(\mathbf{i}A\mathbf{i}A^{-1}) & Tr(\mathbf{i}A\mathbf{j}A^{-1}) & Tr(\mathbf{i}A\mathbf{k}A^{-1}) \\ Tr(\mathbf{j}A\mathbf{i}A^{-1}) & Tr(\mathbf{j}A\mathbf{j}A^{-1}) & Tr(\mathbf{j}A\mathbf{k}A^{-1}) \\ Tr(\mathbf{k}A\mathbf{i}A^{-1}) & Tr(\mathbf{k}A\mathbf{j}A^{-1}) & Tr(\mathbf{k}A\mathbf{k}A^{-1}) \end{pmatrix}$

where, for a quaternion $\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, $Tr(\mathbf{q}) = q_0$.

It is evident that $\phi(A) = \phi(-A)$ so the mapping is 2-to-1. $SU(2)$ is referred to as a *double covering* of $SO(3)$.

By direct calculation we can show that

$$\phi(A) = \phi(a\mathbf{1} + b\mathbf{i} - c\mathbf{j} + d\mathbf{k})$$

$$= -\begin{pmatrix} (c^2+d^2) - (a^2+b^2) & 2(ad+bc) & 2(ac-bd) \\ -2(ad-bc) & (b^2+d^2) - (a^2+c^2) & 2(ab+cd) \\ -2(ac+bd) & 2(cd-ab) & (c^2+b^2) - (a^2+d^2) \end{pmatrix}$$

which is orthogonal and therefore in $O(3)$. By direct calculation we can also verify that $Det(\phi(A)) = 1$ and therefore $\phi(A) \in SO(3)$.

To find the kernel, $ker(\phi) = \{A : \phi(A) = I\}$, we set

$$\phi(A) = \phi(a\mathbf{1} + b\mathbf{i} - c\mathbf{j} + d\mathbf{k}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives us the nine equations

$$\begin{aligned}
ad + bc &= 0 && (1)\\
-ad + bc &= 0 && (2)\\
ac - bd &= 0 && (3)\\
-ac - bd &= 0 && (4)\\
ab + cd &= 0 && (5)\\
-ab + cd &= 0 && (6)\\
(c^2 + d^2) - (a^2 + b^2) &= -1 && (7)\\
(b^2 + d^2) - (a^2 + c^2) &= -1 && (8)\\
(c^2 + b^2) - (a^2 + d^2) &= -1 && (9)
\end{aligned}$$

which has the pair of solutions $(a, b, c, d) = (\pm 1, 0, 0, 0)$.

So, by direct calculation we see that

$\phi(\mathbf{1}) = \phi(-\mathbf{1}) = I$ so $Ker(\phi) = \{\mathbf{1}, -\mathbf{1}\}$.

If we identify the antipodes $A \sim -A$ in $SU(2)$, the homomorphism becomes an isomorphism and then $SU(2)/\mathbf{Z}_2 \cong SO(3)$.

Mathematically, the unification of the electromagnetic and weak forces is given by the group $U(1) \times SU(2)$. In the Standard Model, spontaneous symmetry breaking gives rise to the three vector bosons $W^{\pm}$, Z and the photon γ.

$U(1) \times SU(2) = \{e^{it} e^{(x\mathbf{i} - y\mathbf{j} + z\mathbf{k})} : (t, x, y, z) \in \mathbf{R}^4\}$

So, the generators of $U(1) \times SU(2)$ are $\{i\mathbf{1}, \mathbf{i}, -\mathbf{j}, \mathbf{k}\}$ resulting in the multiplicative structure

$\times$	$i\mathbf{1}$	$\mathbf{i}$	$-\mathbf{j}$	$\mathbf{k}$
$i\mathbf{1}$	$-\mathbf{1}$	$i\mathbf{i}$	$-i\mathbf{j}$	$i\mathbf{k}$
$\mathbf{i}$	$i\mathbf{i}$	$-\mathbf{1}$	$-\mathbf{k}$	$-\mathbf{j}$
$-\mathbf{j}$	$-i\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$
$\mathbf{k}$	$i\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$

If we reverse the sign of only the spatial components $\phi : (i\mathbf{1}, \mathbf{i}, -\mathbf{j}, \mathbf{k}) \mapsto (i\mathbf{1}, -\mathbf{i}, \mathbf{j}, -\mathbf{k})$ we get the resulting structure

$\times$	$i\mathbf{1}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{k}$
$i\mathbf{1}$	$-\mathbf{1}$	$-i\mathbf{i}$	$i\mathbf{j}$	$-i\mathbf{k}$
$-\mathbf{i}$	$-i\mathbf{i}$	$-\mathbf{1}$	$-\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$i\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$
$-\mathbf{k}$	$-i\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$

We observe that ϕ it is not isomorphic since $\phi(i\mathbf{1})\phi(\mathbf{i})\phi(\mathbf{k}) = i(-\mathbf{i})(-\mathbf{k}) = -i\mathbf{j}$ and $\phi(i\mathbf{ik}) = i\phi(-\mathbf{j}) = i\mathbf{j}$ so parity (P) reversal is not invariant for the electroweak interaction. We see that parity reversal also reverses the charge $(\mathbf{i} \to -\mathbf{i}, -\mathbf{j} \to \mathbf{j}, \mathbf{k} \to -\mathbf{k})$. Z is neutral so charge reversal has no effect. So, we also have charge-parity (CP) not invariant. So, CP violation should be expected to be observed in electroweak interactions.

However, if we add time reversal, $\psi : (i\mathbf{1}, \mathbf{i}, -\mathbf{j}, \mathbf{k}) \mapsto (-i\mathbf{1}, -\mathbf{i}, \mathbf{j}, -\mathbf{k})$ restores the original structure

$\times$	$-i\mathbf{1}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{k}$
$-i\mathbf{1}$	$-\mathbf{1}$	$i\mathbf{i}$	$-i\mathbf{j}$	$i\mathbf{k}$
$-\mathbf{i}$	$i\mathbf{i}$	$-\mathbf{1}$	$-\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-i\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$
$-\mathbf{k}$	$i\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$

and $\psi(i\mathbf{1})\psi(\mathbf{i})\psi(\mathbf{k}) = (-i\mathbf{1})(-\mathbf{i})(-\mathbf{k}) = i\mathbf{j}$ and $\psi(i\mathbf{ik}) = \psi(-i\mathbf{j}) = i\mathbf{j}$. This corresponds to charge-parity-time (CPT) invariance in the electroweak interaction.

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