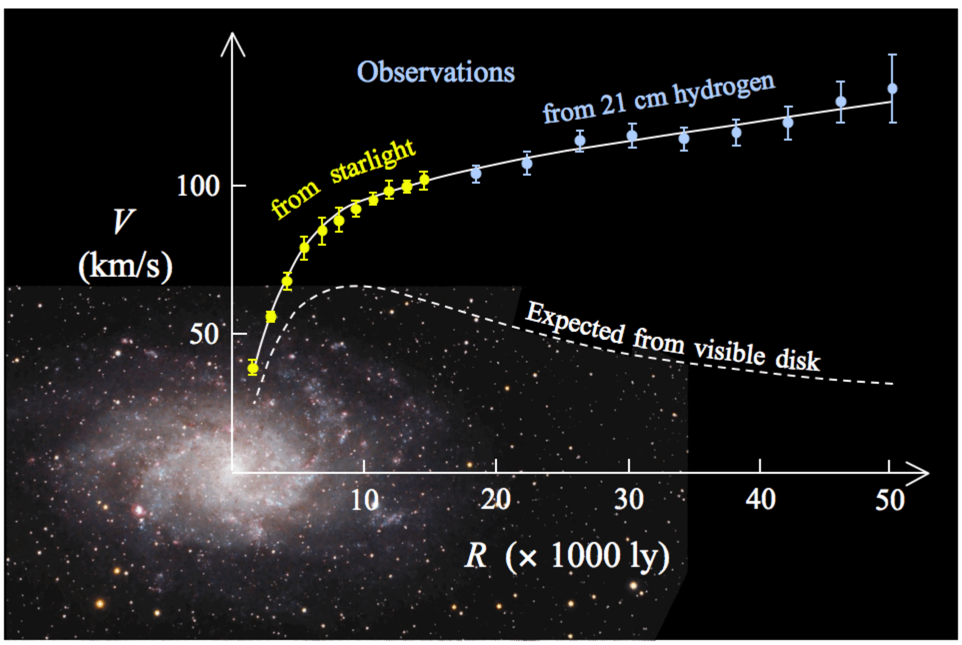
Galaxy Rotation

Summary: There is an anomaly in galaxy rotation insofar that stars do not appear to obey Kepler rotation in their orbits around the galactic center. For many galaxies, beyond a fixed radius, the orbital speed is not dependent on the distance from the center but approximates a constant. In the following we assess whether the anomaly can be explained using the Kerr metric solution and conclude that neither dark matter nor modified gravity are required.

Galactic Rotation Rate:

It has been observed that galaxy rotation deviates from expected Kepler rotation as in the following figure.



Let us consider a disk whose rate of rotation is not rigid but allows for different rates depending on the distance from the center. Suppose the mass density has a constant value ρ.

Consider the annulus of width dr and radius r. Its mass and moment of inertia are $2\pi\rho r dr$ and $2\pi\rho r^3 dr$ respectively. Let $\omega(r)$ be the angular velocity of the disk at r. Then the total angular momentum of the disk of total radius R is $L = 2\pi\rho\int_0^R \omega(r)r^3dr$. If the density is not constant but only depends on r then $L = 2\pi\int_0^R \rho(r)\omega(r)r^3dr$

The Kerr Metric:

The Kerr metric has been discussed in the articles *Quaternion Spacetime* and *The Kerr Metric*. It is the axially symmetric solution to the GR field equations around a rotating gravitating body with angular momentum J:

$$ds^2 = (1 - \alpha/\tilde{R})dt^2 + \frac{2\alpha a sin^2\theta}{\tilde{R}}dtd\phi - \frac{\Sigma}{\Delta}dR^2 - d\Omega^2$$

where c is set at 1, t is the elapsed time of a clock 'at infinity', r is the scalar distance, $R = (r^3 + \alpha^3)^{1/3}$, $\alpha = 2GM$, $\tilde{R} = \frac{R^2+a^2cos^2\theta}{R}$, $a = J/M$, $\Sigma = R^2 + a^2cos^2\theta$, $\Delta = R^2 - \alpha R + a^2$, and

$$d\Omega^2 = \Sigma d\theta^2 + (R^2 + a^2 + \frac{\alpha a^2 sin^2\theta}{\tilde{R}})sin^2\theta d\phi^2$$

The Symmetric Bilinear Form for the Kerr Metric

The Kerr metric can be written in bilinear form as $\begin{pmatrix} dt \\ dR \\ d\theta \\ d\phi \end{pmatrix}^T K \begin{pmatrix} dt \\ dR \\ d\theta \\ d\phi \end{pmatrix}$

where $K = \begin{pmatrix} (1-\frac{\alpha}{\tilde{R}}) & 0 & 0 & \frac{\alpha a sin^2\theta}{\tilde{R}} \\ 0 & -\frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & -\Sigma & 0 \\ \frac{\alpha a sin^2\theta}{\tilde{R}} & 0 & 0 & -(R^2 + a^2 + \frac{\alpha a^2}{\tilde{R}}sin^2\theta)sin^2\theta \end{pmatrix}$

which we can write as $K = \begin{pmatrix} A & 0 & 0 & E \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ E & 0 & 0 & D \end{pmatrix}$

The eigenvalues for K are $\lambda = \frac{(A+D)\pm\sqrt{(A-D)^2+4E^2}}{2}$,$B$,$C$.

Then the diagonal bilinear form for the Kerr metric is

$$Q^T K Q = \begin{pmatrix} \frac{(A+D)+\sqrt{(A-D)^2+4E^2}}{2} & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & \frac{(A+D)-\sqrt{(A-D)^2+4E^2}}{2} \end{pmatrix}$$

where Q is orthogonal since K is real symmetric

and where $\begin{pmatrix} dt \\ dR \\ d\theta \\ d\phi \end{pmatrix} = Q \begin{pmatrix} dt' \\ dR' \\ d\theta' \\ d\phi' \end{pmatrix}$ is the orthogonal coordinate transformation between the two sets of coordinates.

So, the Kerr metric in canonical (diagonal) form is

$$ds^2 = \frac{(A+D)+\sqrt{(A-D)^2+4E^2}}{2} dt'^2 - |B| dR'^2 - |C| d\theta'^2 - \left| \frac{(A+D)-\sqrt{(A-D)^2+4E^2}}{2} \right| d\phi'^2.$$

For a unit mass in orbit around the central gravitating body,

$$1 = \frac{(A+D)+\sqrt{(A-D)^2+4E^2}}{2} E_{rg}^2 - |B| P_{R'}^2 - |C| P_{\theta'}^2 - \left| \frac{(A+D)-\sqrt{(A-D)^2+4E^2}}{2} \right| P_{\phi'}^2.$$

where E_{rg} is the energy. The canonical coordinates are not static but rather the frame rotates and the rate of rotation is dependent on R, a, and α (such dependence produces the spiral arms). For a rotating galaxy $a = L$ as in the disk discussed above. We will now assume that motion is circular in the equatorial plane so the above equation becomes

$$1 = \frac{(A+D)+\sqrt{(A-D)^2+4E^2}}{2} E_{rg}^2 - \left| \frac{(A+D)-\sqrt{(A-D)^2+4E^2}}{2} \right| P_{\phi'}^2.$$

We suggest the hypothesis that with respect to the canonical coordinates, the motion is actually Kepler motion identical to the case where

$$1 = (1 - \frac{\alpha}{R}) E_{rg}^2 - R^2 sin^2 P_{\phi}^2$$

but does not appear so because of the rotation of the canonical coordinate frame itself.** The second to last relation is between R,a, and α. The latter $\alpha = 2GM$ so the relation in each is between R, $a = 2\pi \int_0^R \rho(r)\omega(r) r^3 dr$, and $M = 2\pi \int_0^R \rho(r) r dr$.* We can expect the observed rotation to deviate from the computed Kepler rotation by an amount equal to a rotation in the canonical coordinate frame itself where $d\phi/d\tau = \frac{1}{\kappa}\left[\frac{\alpha a}{R}\right]$ and where

$$\kappa = Det \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}$$

Conservation of Total Galactic Angular Momentum:

Let R_∞ be large enough so that $\rho(r) \to 0$ as $r \to R_\infty$. Then the conservation of galactic angular momentum implies $2\pi \int_0^{R_\infty} \rho(r)\omega(r)r^3 dr$ is constant. If the mass distribution shifts outward or inward toward the center, the distribution of angular velocity $\omega(r)$ must shift accordingly.

Suppose that initially a galaxy displays something approximating Kepler motion. That is, suppose $\omega(r)$ increases as $r \to 0$. Then, the result of mass falling inward, perhaps due to the action of a black hole (defined to be an object whose mass is compressed within 1.5 Schwarzschild radii), must be a shift outward in the ω distribution. Over time, this would result in galactic motion appearing less Kepler with respect to the static Kerr coordinates.

Footnote *:

Both $\rho(r)$ and $\omega(r)$ might also depend on ϕ but we are here using $\rho(r) = \int_0^{2\pi} \rho(r,\phi)d\phi$ and $\omega(r) = \int_0^{2\pi} \omega(r,\phi)d\phi$.

Footnote **:

In traditional black hole astrophysics the upper bound for the central angular momentum $a \le GM = \alpha/2$. In the article *The Schwarzschild Metric* we show that by viewing space around a black hole as having negative curvature we can remove the coordinate singularities leaving only the physical singularity at the center, assuming the central mass is in fact compressed to a single point. It is more likely to be compressed to some maximum attainable density. See also *The Kerr Metric*. That implies there is no inherent upper bound for a. So, we can apply the Kerr metric to a large structure such as a galaxy.

Let $\mathbf{u}$ be a 4-vector and $-\epsilon = g_{tt}u^t + g_{t\phi}u^\phi$ and $l = g_{t\phi}u^t + g_{\phi\phi}u^\phi$.

We can solve for $u^{\phi} = d\phi/d\tau = \frac{1}{\kappa}\left[(1-\frac{\alpha}{R})l + \frac{\alpha a}{R}\epsilon\right]$ where

$$\kappa = Det\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}$$

For the static Kerr coordinates, a mass with no initial angular momentum and radial motion only, can acquire angular velocity. Setting initial $l = 0$ and $\epsilon = 1$ we have $d\phi/d\tau = \frac{1}{\kappa}\left[\frac{\alpha a}{R}\right]$. W.r.t. canonical coordinates, $-\epsilon' = g_{t't'}u^{t'}$ and $l' = g_{\phi'\phi'}u^{\phi'}$ so $d\phi'/d\tau = 0$. Therefore, the canonical frame is rotating relative to the static frame.

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