The Hubble Parameter

The cosmological equations are:

$$\begin{aligned}
\frac{8\pi G\rho}{3} &= \frac{k}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 && (1)\\
8\pi G p &= -2\frac{\ddot{a}}{a} - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 && (2)\\
\frac{8\pi G}{3}(\rho + 3p) &= -2\frac{\ddot{a}}{a} && (3)
\end{aligned}$$

In equation (1) and (2), $k = -1, 0, 1$ corresponding to the tri-curvature of negative, flat and positive. The Hubble parameter $H = \frac{\dot{a}}{a} = \frac{1}{t_H}$. The units of the Hubble parameter are in s^{-1} allowing us to define Hubble time (now) as $t_H = H_0^{-1}$ so $H_0 = t_H^{-1}$ is true by definition.

Models based on the assumption that $p = 0$ and $\rho a^3 = constant$ for the three curvature values were developed by Friedman in the 1920s.*

Setting $A = \frac{8\pi G\rho a^3}{3} = constant$ within each solution, the three Friedman solutions are:

- For $k = 0$: $a = a_0 t^{2/3}$ for $0 < t$. This gives $\dot{a} = \frac{2}{3}a_0 t^{-1/3}$ and $H = \frac{2}{3t}$ and $4a_0^3 = 9A$. Note that $H \to 0$ as $t \to \infty$
- For $k = +1$: $t = \frac{1}{2}A(\theta - sin\theta)$ and $a = \frac{1}{2}A(1 - cos\theta)$ for $0 < \theta < 2\pi$. Then $\dot{a} = \frac{sin\theta}{1-cos\theta}$ and $H = \frac{sin\theta}{\frac{1}{2}A(1-cos\theta)^2}$
- For $k = -1$: $t = \frac{1}{2}A(sinh\eta - \eta)$ and $a = \frac{1}{2}A(cosh\eta - 1)$ for $0 < \eta$. Then $\dot{a} = \frac{sinh\eta}{cosh\eta-1}$ and $H = \frac{sinh\eta}{\frac{1}{2}A(cosh\eta-1)^2}$. Note that $H \to 0$ as $\eta \to \infty$.

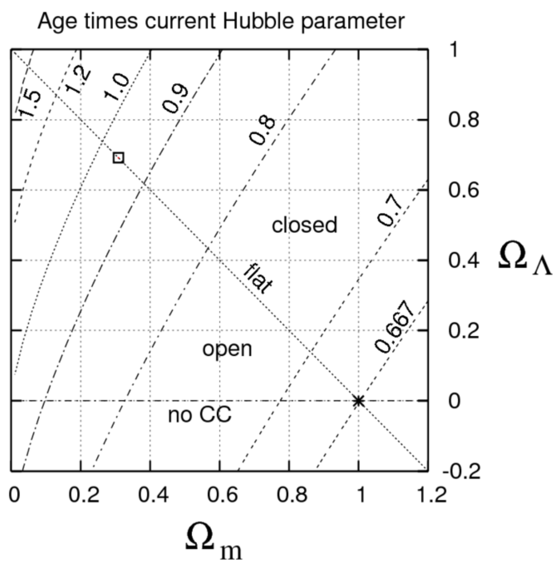
We can estimate H_0 empirically and as we have seen we can define the Hubble time $t_H = H_0^{-1}$. We would like to know how the actual time $t > 0$ compares to H^{-1}.

For $k = 0$: $t/H^{-1} = 2/3$.

For $k = +1$: $t/H^{-1} = \frac{sin\theta(\theta - sin\theta)}{(1-cos\theta)^2} < 2/3$ and diverges $\downarrow -\infty$ as $\theta \to 2\pi$

For $k = -1$: $t/H^{-1} = \frac{sinh\eta(sinh\eta - \eta)}{(cosh\eta - 1)^2} > 2/3$ and converges $\uparrow 1$ as $\eta \to \infty$.

So, only for $k = -1$ is H^{-1} eventually a good approximation for t.



We can see from the graph above that for $\Omega_\Lambda = 0$ and $\rho = .04\rho_c$, $t/t_H \approx .96$ and therefore $t = 13.8$Gyr. This assumes $t_H \approx 14.4$Gyr.

$t/t_H \approx .96$ corresponds to $\eta \approx 5.0$. We can then calculate

$$A = 2t/(sinh\eta - \eta)$$

We get $A = 2(13.8 \times 10^9 y)/69.2 = 1.26 \times 10^{16} s$. Consequently,

$$a = \frac{1}{2}A(cosh\eta - 1) = 4.61 \times 10^{17} s$$

Using the value $H_0 = 2.2 \times 10^{-18} s^{-1}$ we can calculate

$$\dot{a} = aH_0 = (4.6 \times 10^{17} s)(2.2 \times 10^{-18} s^{-1}) = 1.0$$

This is in close agreement with $\dot{a} = \frac{sinh\eta}{cosh\eta - 1} = 1.01$ for $\eta = 5$.

The Deceleration Parameter

$$\dot{H} = \frac{a\ddot{a} - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{\dot{a}^2}{a^2}(\frac{\ddot{a}a}{\dot{a}^2} - 1) = -H^2(1+q)$$

where we define the *deceleration parameter* $q \equiv -\frac{\ddot{a}a}{\dot{a}^2}$. In the hyperbolic space under consideration, $q = \frac{1}{2}A\frac{(cosh\eta - 1)^2}{sinh^2\eta} \downarrow 0$ as $\eta \to \infty$.

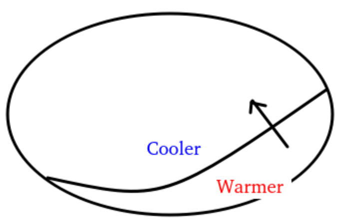
For small q, $\frac{\dot{H}}{H} \approx -\frac{\dot{a}}{a}$

The CMB

In the Cosmic Microwave Background we observe small variations in the temperature. The Universe, as a 3-pseudosphere, has the spatial metric

$$d\sigma^2 = d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)$$

It can be expected to have variations in its expansion rate along different directions, the greatest being in the χ direction. This will result in the wavelength of photons being stretched more in that direction lowering the temperature of the black body radiation. The arrow in the figure below shows the χ gradient accompanied by decreasing temperature in the CMB.



Red Shift

We define *red shift* z as $z = \frac{\lambda_0}{\lambda_e} - 1$ where λ_0 is an observed wavelength and λ_e is the emitted wave length.

Proper red shift can occur as a Doppler effect due to the motion of an object in space whereas cosmological red shift is due to the expansion of the Universe.

We can interpret cosmological red shift in either of two fundamentally equivalent ways. It can be viewed as either due to the expansion of space or the dilation of time. We will assume $k = -1$.

In the former, we set $a = \frac{1}{2}A(cosh\eta - 1)$ and assume $z + 1 = \frac{a_0}{a_1}$ where a_0 is the scale factor of the Universe now and a_1 is that at some time in the past. Then $z + 1 = \frac{cosh\eta_0 - 1}{cosh\eta_1 - 1}$ where η_0 and η_1 correspond to now and the designated time in the past.

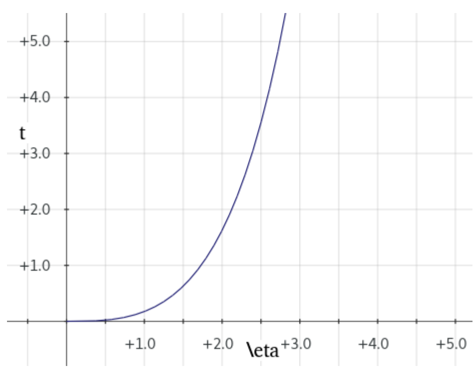
In the latter case, $\frac{dt}{d\eta} = \frac{1}{2}A(cosh\eta - 1)$. Then $\frac{dt_0/d\eta_0}{dt_1/d\eta_1} = \frac{cosh\eta_0 - 1}{cosh\eta_1 - 1} = z + 1$.

The Tempo of Time

The *expansion parameter* $\eta = \int \frac{dt}{a}$ and consequently

$$\eta_0 = \int_0^{t_0} \frac{dt}{a} = \int_0^{\eta(t_0)} d\eta$$

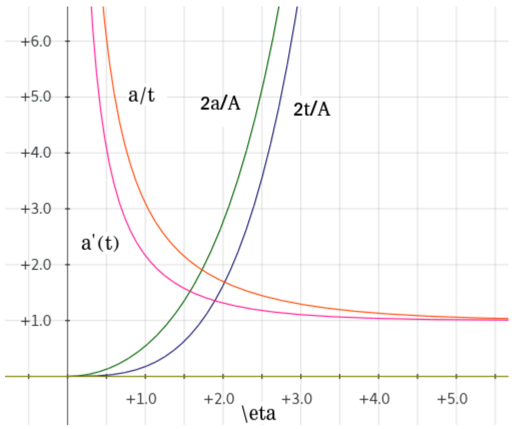
The following graph shows the relation between t and η.

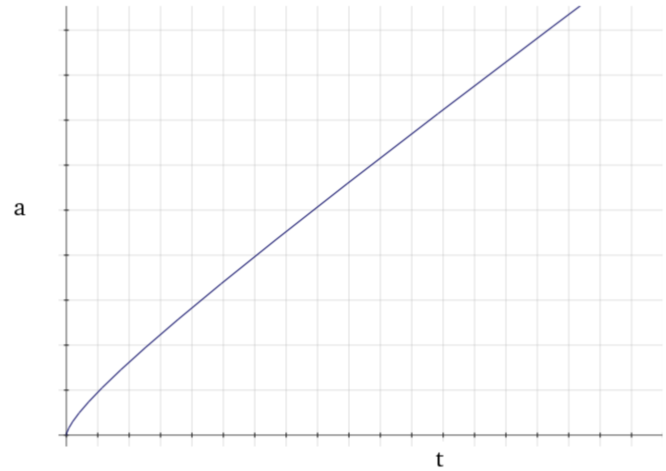


We can see that the tempo of time is near zero when $\eta \approx 0$ and speeds up with increasing η.

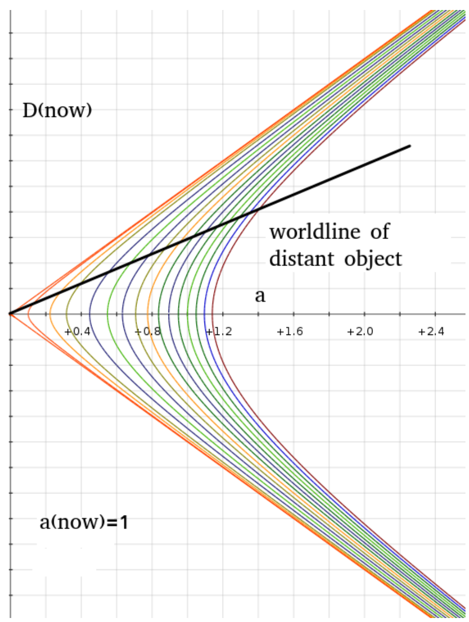
We can calculate $\dot{a} = \frac{da}{d\eta}\frac{d\eta}{dt}$. Now, $\frac{da}{d\eta} = \frac{1}{2}Asinh(\eta)$ and from $\eta = \int \frac{dt}{a}$ we have $\frac{d\eta}{dt} = \frac{1}{a}$. Consequently, $\dot{a} = \frac{sinh\eta}{cosh\eta - 1}$ (as calculated above).

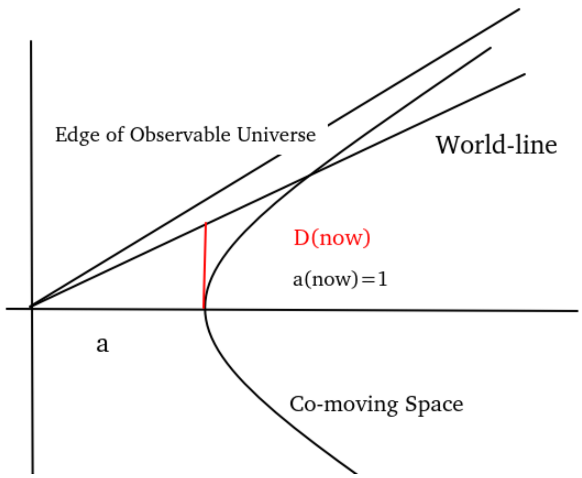
The following graph shows $\dot{a} = a'(t)$, a/t, a, and t all w.r.t. η.





Galactic Distance and Recession Velocity





Suppose a photon leaves a distant galaxy at time t_1 and arrives at an observer at time t_0, the *distance now* of the distant galaxy is

$$D_{now} = a(t_0)\int_{t_1}^{t_0} \frac{dt}{a(t)} = a(t_0)[\eta(t_0) - \eta(t_1)] = a(t_0)[\eta_0 - \eta_1]$$

Then

$$\dot{D}_{now} = \dot{a}(t_0)[\eta(t_0) - \eta(t_1)] + a(t_0)[\frac{d\eta}{dt_0} - \frac{d\eta}{dt_1}\frac{dt_1}{dt_0}]$$

and upon substitution we get

$$\dot{D}_{now} = \dot{a}(t_0)[\eta(t_0) - \eta(t_1)] + a(t_0)[\frac{2}{A(cosh\eta_0 - 1)} - \frac{2}{A(cosh\eta_0 - 1)}]$$

So $v = \dot{a}(t_0)[\eta_0 - \eta_1]$ is the hypothesized current recession velocity but cannot be observed until emitted photons arrive with red shift

$$z = \frac{cosh(\eta_0 + \eta_0 - \eta_1) - 1}{cosh(\eta_0) - 1} - 1$$

where $2\eta_0 - \eta_1$ is the value of η at time of arrival.

Since $\eta_0 - (\eta_0 - \eta_1) - [\eta_1 - (\eta_0 - \eta_1)] = \eta_0 - \eta_1$ we have

the recession velocity now $v_0 = \frac{sinh\eta_0}{cosh\eta_0 - 1}(\eta_0 - \eta_1)$ and

the recession velocity at time of emission $v_1 = \frac{sinh\eta_1}{cosh\eta_1 - 1}(\eta_0 - \eta_1)$.

The *Hubble law* states that

$v_{now} = D_{now}H_{now}$

$= a(t_0)(\eta_0 - \eta_1)H_{now} = a(t_0)(\eta_0 - \eta_1)\frac{\dot{a}(t_0)}{a(t_0)} = \dot{a}(t_0)(\eta_0 - \eta_1)$.

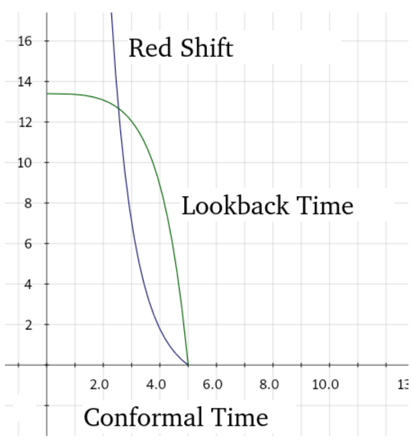
Example: Suppose we observe a galaxy with $z = 8.0$. Assume $\eta_0 = 5.0$, $t_0 = 13.8$Gyr, and $a(t_0) = 14.2 \times 10^9$.

Then $\frac{cosh5.0-1}{cosh\eta_1-1} = 9.0$. We solve for $\eta_1 = 2.9$ and determine that $D_{now} = a(t_0)(\eta_0 - \eta_1) = 14.2(5.0 - 2.9) \times 10^9 = 29.8 \times 10^9 lyr = 29.8Glyr$. A current red shift of $z = 8.0$ corresponds to a recession velocity of 2.3c when the light was emitted and 2.1c now.

The duration of light travel is $\tau = t_0 - t_1$ where $\frac{t_0}{t_1} = \frac{sinh\eta_0-\eta_0}{sinh\eta_1-\eta_1} = \frac{sinh5.0-5.0}{sinh2.9-2.9} = 11.2$.

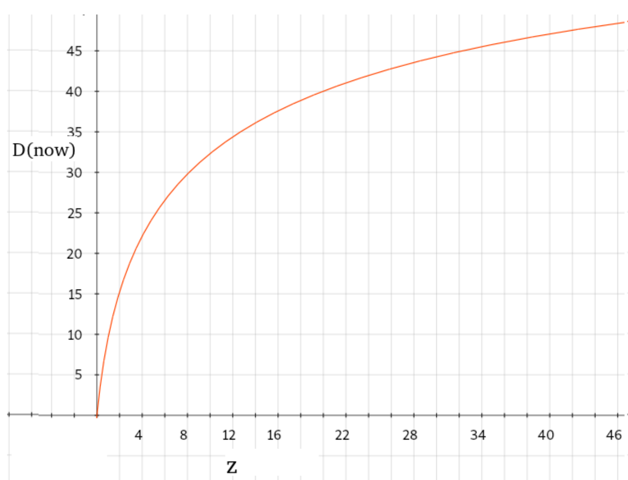
Then $\tau = t_0 - t_0(\frac{t_1}{t_0}) = t_0 - \frac{t_0}{11.2} = 12.6Gyr$.**

We can graph the relationship between red shift and lookback time based on conformal time:



We may consider the observed distance to be the lookback time (τ) multiplied by the speed of light. The observed distance is not the same as D_{now}, the current distance, since the Universe has expanded since light began its journey from the observed object.

The above graph shows the relationship of red shift and D_{now} both dependent on conformal time η. The scale on the left is in redshift units but also Billion Light Years. Corresponding to $z = 1$ we have $D_{now} = 9.19Glyr$ and corresponding to $z = 4$ we have $D_{now} = 20.75Glyr$. In each case the two values have a common η value. Not show on the graph is that $D_{now} \to 71Glyr$ as $z \to \infty$. We can also parameterize the two graphs dependent on conformal time η:



Note that for any $0 < \eta_1 < \eta_0$, $\tau < t_0$ (the current age of the Universe) and as $\eta_1 \to 0$, $\tau \to t_0$. This eliminates the so-called horizon problem. No matter how far apart two entities are now, by taking a as small as we like, the two entities can be made arbitrarily close.

We should also note the following. We showed above that $\eta_0 - \eta_1$ remains contant as the Universe expands. As η_0 increases new objects will come into view as they will appear to emerge from the hyper-surface of last scattering.

The above calculation shows that the light from such an object has time within the current age of the Universe to reach us even though it is receding faster than the speed of light. Using the above method ($\eta_1 = 4.3$) we can show that D_{now} for $z = 1$ is about $9.9\,Glyr$. That is, galaxies with $z = 1$ have receded from us out to that distance though their light has taken *only* $7.0\,Gyr$ to reach us. D_{now} for $z = 1$ was the Hubble distance ($D = ct_H$) $7.0\,Gyr$ ago. Galaxies on the Hubble sphere ($D_{now} = ct_H$) are now receding from us at about $1.0c$. Light emitted now will reach us in $14.4\,Gyr$ ($= t_H$)with a red shift of $z = \frac{cosh(5.7)-1}{cosh(5.0)-1} - 1 \approx 1.0$.

Since $\dot{a}(t_0) \downarrow 1$ in the case of negative cosmological curvature we have as a limiting case $v = \eta_0 - \eta_1$.

Note that between our galaxy and some other, $t_0 - t_1$ is not constant as η increases but $x + (\eta_0 - \eta_1) - x = \eta_0 - \eta_1 = constant$. So, the red shift of a distant galaxy will converge to $z = lim_{x\to\infty} \frac{cosh(x+\eta_0-\eta_1)-1}{cosh(x)-1} - 1 = e^{\eta_0 - \eta_1} - 1$ as the Hubble parameter $\to 0$ while the recession velocity tends to $v = \eta_0 - \eta_1$.

Null Paths

For the following we use the generalized Robertson-Walker coordinates.

$$ds^2 = dt^2 - \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right]$$

Let $\gamma = \gamma(s)$ be the path of a photon through space-time parameterized by space-time arclength s. The scale factor is assumed to be $a = 1$.

Then

$$\frac{d\gamma}{ds} = \frac{dt}{ds}\partial_t + \frac{dr}{ds}\partial_r + \frac{d\theta}{ds}\partial_\theta + \frac{d\phi}{ds}\partial_\phi$$

Now, $\langle\partial_t, \partial_t\rangle = 1$, $\langle\partial_r, \partial_r\rangle = \frac{1}{1-kr^2}$, $\langle\partial_\theta, \partial_\theta\rangle = r^2$, and $\langle\partial_\phi, \partial_\phi\rangle = r^2 sin^2\theta$

Then

$$0 = \langle \frac{d\gamma}{ds}, \frac{d\gamma}{ds} \rangle = dt^2 - \frac{dr^2}{1-kr^2} - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2$$

We are interested in paths terminating along a single line of sight so

$$d\theta = d\phi = 0$$

Then

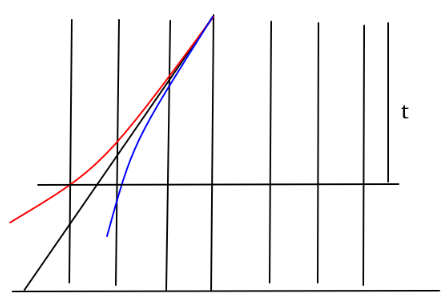
$$dt^2 = \frac{dr^2}{1-kr^2}$$

We have three cases:
For $k = 0$, $r = \rho$. For $k = +1$, $r = sin\beta$. For $k = -1$, $r = sinh\chi$

Then for the three cases, $dt = d\rho$, $dt = d\beta$, and $dt = d\chi$.

For flat space we get a cone as in special relativity. The other two cases are not cone-like but curved.

Assuming $\beta = \chi = 0$ when $\rho = 0$ we can say $\rho = \beta = \chi$. Then $sin\beta < \rho < sinh\chi$ and we have the following picture:



The above graph shows world-lines in co-moving co-ordinates. Also shown are the null paths followed by photons in the tri-curvature.

Red= $sinh\chi$; Black= ρ; and Blue= $sin\beta$

The above picture is scalable so for a specific scale factor a we have $asin\beta < a\rho < asinh\chi$.

We note above that $r = sinh\chi$ and so $\frac{dr}{dt} = cosh\chi\frac{d\chi}{dt} > 1$ for $\chi > 0$. Then the speed of light w.r.t. Robertson-Walker time is greater than 1. This is not unusual in Relativity Theory but it does not indicate that when c is measured locally it is different from its invariant value.

Consider a distant galaxy with cosmological redshift z. We let our unit of distance to be the wavelength of a specific spectral line of Kr-86 and our unit of time to be based on the frequency of a specific spectral line of Cs-133. Then our unit of length is λ_{Kr} and our unit of time is $1/\nu_{Cs}$. Chose m, n s.t. $\frac{m\lambda_{Kr}}{n(1/\nu_{Cs})} = c$. Then $\frac{m}{n}\lambda_{Kr}\nu_{Cs} = c$. We shall assume this definition of the speed of light can be transported backwards in time to other galaxies.

Due to the expansion of the Universe we have $z+1 = \frac{\lambda}{\lambda'} = \frac{\nu'}{\nu}$ where the primes represent the values in the distant galaxy.

Then $\lambda\nu = \lambda'\nu'$ and consequently, $\frac{m}{n}\lambda_{Kr}\nu_{Cs} = \frac{m}{n}\lambda'_{Kr}\nu'_{Cs} = c$. This implies the speed of light is measured the same (w.r.t. proper time τ) at the distant galaxy. It follows that $\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = 1$. Then $\frac{dt}{d\tau} = \frac{1}{cosh\chi}$.

A somewhat more elegant proof without the assumption made above is as follows:

The metric $ds^2 = dR^2 - R^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$ describes a four dimensional flat space. It is similar to the Robertson-Walker metric $ds^2 = dt^2 - R^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$ which describes a curved four dimensional space-time. The difference is dR is replaced by dt in the first expression.

Define the Special Relativity coordinates of an observer at χ to be:

$$T = Rcosh\chi \quad (4)$$
$$X = Rsinh\chi cos\theta \quad (5)$$
$$Y = Rsinh\chi sin\theta cos\phi \quad (6)$$
$$Z = Rsinh\chi sin\theta sin\phi \quad (7)$$

Then $R^2 = T^2 - X^2 - Y^2 - Z^2$. It has the metric of the above four dimensional flat space.

The above quadratic written as

$$\begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix}$$

is invariant under every Lorentz transformation Λ

$$\begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix}^T \Lambda^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix} = R^2$$

with R^2 as an invariant. Furthermore,

$$\begin{aligned} 0 &= dT'^2 - dX'^2 - dY'^2 - dZ'^2 \\ &= \left[\Lambda \begin{pmatrix} dT \\ dX \\ dY \\ dZ \end{pmatrix}\right]^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} dT \\ dX \\ dY \\ dZ \end{pmatrix} \\ &= \begin{pmatrix} dT \\ dX \\ dY \\ dZ \end{pmatrix}^T \Lambda^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} dT \\ dX \\ dY \\ dZ \end{pmatrix} \\ &= \begin{pmatrix} dT \\ dX \\ dY \\ dZ \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} dT \\ dX \\ dY \\ dZ \end{pmatrix} \\ &= dT^2 - dX^2 - dY^2 - dZ^2 = 0 \end{aligned}$$

So, the speed of light will also be invariant in this space-time. So, all observers see the speed of light

$$c = \frac{dX^2 + dY^2 + dZ^2}{dT^2} = 1$$

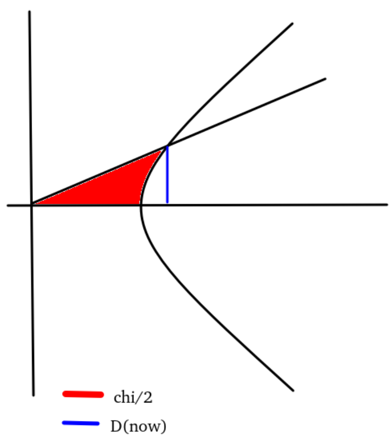
.

How Big Is the Observable Universe?

For hyperbolic space ($k = -1$) the co-moving metric

$$d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)$$

is unbounded (for all $t > 0$) since there is no upper bound for χ. However, the maximum value for D_{now} is $D_{now} = a(t_0)(\eta_0 - 0) = 14.2(5.0 - 0) \times 10^9 = 71.0 \times 10^9 lyr = 71.0 Glyr$.



Consider the foliation of co-moving spaces indexed by the scale factor a. The world-line of an object intersects the unique past co-moving space at some $t_1 < t_0$ in our past (as shown above) such that

$$a(t_1)sinh\chi = D_{now} = a(t_0)(\eta_0 - \eta_1)$$

and

$$a(t_1) = \frac{a(t_0)(\eta_0 - \eta_1)}{sinh\chi}$$

To calculate the current space-time metric of $\{(t, \chi, \theta, \phi) : \chi = constant\}$ we determine the value $t_1 < t_0$ where one such world-line intersects the past co-moving space as above. Then with $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$,

$$ds^2 = dt^2 - a^2(t_1)(d\chi^2 + sinh^2\chi d\Omega^2)$$

Along a world-line $\chi = constant$ so $d\chi = 0$ and we previously calculated

$$a(t_1) = \frac{a(t_0)(\eta_0 - \eta_1)}{sinh\chi} = \frac{D_{now}}{sinh\chi}$$

Then $ds^2 = dt^2 - D_{now}^2 d\Omega^2$

We note that $ds^2 \neq 0$ since null paths do not coincide with world-lines.

We can calculate the co-moving coordinate χ without much difficulty. Assume $t_1 < t_0$. From before we calculated $a(t_1) = \frac{a(t_0)(\eta_0 - \eta_1)}{sinh\chi}$ Then

$$sinh\chi = \frac{a(t_0)}{a(t_1)}(\eta_0 - \eta_1)$$

and

$$sinh\chi = \frac{cosh\eta_0 - 1}{cosh\eta_1 - 1}(\eta_0 - \eta_1)$$

Then

$$\chi = sinh^{-1}\left[\frac{cosh\eta_0 - 1}{cosh\eta_1 - 1}(\eta_0 - \eta_1)\right]$$

One observation we can make is that $\chi \to \infty$ as $\eta_1 \to 0$.

How Old Was the Universe at Last Scattering?

This was when the plasma phase ended and light first emerged. The microwave background is estimated to have red shift = 1100. This corresponds to $\eta = 0.36$. This corresponds to an age 1.13×10^{-4} times the age of the Universe (13.8×10^9years) which we can therefore compute to be 1.56×10^6 years. This incidentally corresponds to $\chi = 9.24$.

How Fast is the Universe Expanding?

We might suggest that the rate of expansion is expressed in terms of how fast the scale factor a is increasing. Currently $\dot{a} \approx 1.01c$. The upper limit to $v = \dot{a}(t)(\eta_0 - \eta_1)$ is when $\eta_1 = 0$. So the upper limit to the rate of expansion would be $v = \dot{a}(t)\eta_0 = 1.01(5.0) = 5.05c$. Asymptotically (as $\dot{a} \downarrow 1$) this becomes $v = \eta_0$ in units of c. So, the rate of expansion does not approach zero nor become exponential. The rate of expansion will eventually grow linearly proportional to conformal time.

*The formula $\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$ follows from the cosmological equations. Setting $p = 0$ we get

$$\dot{\rho} = -3\rho\frac{\dot{a}}{a}$$

and consequently

$$\frac{\dot{\rho}}{\rho} \propto \frac{\dot{a}}{a}$$

We can show that $p = 0$ implies $\rho a^3 = constant$ using the formula

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$$

Setting $p = 0$ we have

$$\dot{\rho} = -3(\rho)\frac{\dot{a}}{a}$$

then

$$\dot{\rho} = -3(\rho)\frac{a^2\dot{a}}{a^3}$$

and

$$\dot{\rho}a^3 = -3\rho a^2\dot{a}$$

then

$$\dot{\rho}a^3 + 3\rho a^2\dot{a} = 0$$

Solving this ODE (w.r.t.t) we get

$$\rho a^3 = constant$$

**A more recent example exists of a galaxy GN-z11 with redshift 11.1. Assume $\eta_0 = 5.0$, $t_0 = 13.8$Gyr, and $a(t_0) = 14.2 \times 10^9$.

Then $\frac{cosh 5.0-1}{cosh\eta_1-1} = 12.1$. We solve for $\eta_1 = 2.6$ and determine that $D_{now} = a(t_0)(\eta_0 - \eta_1) = 14.2(5.0 - 2.6) \times 10^9 = 34.0 \times 10^9 lyr = 34.0 Glyr$. A current red shift of $z = 11.1$ corresponds to a recession velocity of 2.6c when the light was emitted and 2.4c now.

The duration of light travel is $\tau = t_0 - t_1$ where $\frac{t_0}{t_1} = \frac{sinh\eta_0-\eta_0}{sinh\eta_1-\eta_1} = \frac{sinh 5.0-5.0}{sinh 2.6-2.6} = 16.9$.

Then $\tau = t_0 - t_0(\frac{t_1}{t_0}) = t_0 - \frac{t_0}{16.9} = 13.0 Gyr$.

This shows that galaxies had started forming within 1 billion years of the Big Bang event.

For GN-z11, the $\Lambda - CDM$ model gives a lookback time of 13.4 Gyr which implies its formation within 400 Myr after the big Bang. In general, hyperbolic space shows a reduced lookback time of 400-500 Myr for high red shift objects.

For high red shift objects recently observed by the JWST set $z = 14$.

Then $\eta_1 = 2.3$ and $\tau = 13.3$Gyr. Such a galaxy would have formed within 500 million years after the Big Bang. $\Lambda - CDM$ implies its beginning almost immediately after recombination.

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