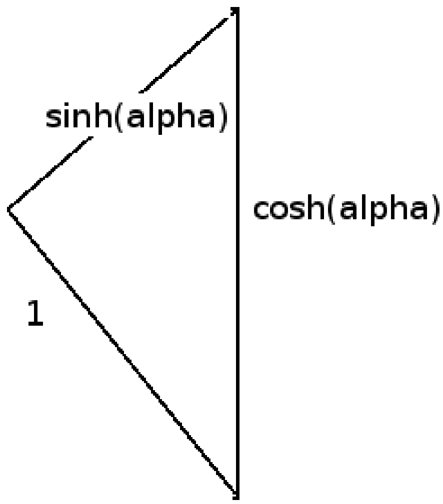
Hyperbolic Dynamics



Summary: Here we describe some of the properties of hyperbolic dynamics such as velocity addition and red shift. Here we develop a metric for the negatively curved Universe and show it is close to our observed metric.

$$\begin{pmatrix} \Delta t_2 \\ \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t_1 \\ \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \end{pmatrix}$$

where $cosh(\alpha) = \frac{1}{\sqrt{1-v^2}}$ and $sinh(\alpha) = \frac{v}{\sqrt{1-v^2}}$. Then $v = tanh(\alpha)$.

Let S, S', S'' be frames moving with uniform velocity along the x-direction. Let S' be moving with velocity v with respect to S and S'' be moving with velocity w with respect to S'. First note that $\frac{sinh(\alpha)}{cosh(\alpha)} = v$ and $\frac{sinh(\beta)}{cosh(\beta)} = w$. Then

$$\begin{pmatrix} \Delta t'' \\ \Delta x'' \\ \Delta y'' \\ \Delta z'' \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cosh(\beta) & sinh(\beta) & 0 & 0 \\ sinh(\beta) & cosh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

.

$$= \begin{pmatrix} cosh(\alpha)cosh(\beta)+sinh(\alpha)sinh(\beta) & cosh(\alpha)sinh(\beta)+sinh(\alpha)cosh(\beta) & 0 & 0 \\ cosh(\alpha)sinh(\beta)+sinh(\alpha)cosh(\beta) & cosh(\alpha)cosh(\beta)+sinh(\alpha)sinh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

$$= \begin{pmatrix} cosh(\alpha+\beta) & sinh(\alpha+\beta) & 0 & 0 \\ sinh(\alpha+\beta) & cosh(\alpha+\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

The combined velocity is

$\frac{sinh(\alpha+\beta)}{cosh(\alpha+\beta)} = \frac{cosh(\alpha)sinh(\beta)+sinh(\alpha)cosh(\beta)}{cosh(\alpha)cosh(\beta)+sinh(\alpha)sinh(\beta)}$ and dividing top and bottom by $cosh(\alpha)cosh(\beta)$ gives the combined velocity $\frac{sinh(\alpha+\beta)}{cosh(\alpha+\beta)} = \frac{v+w}{1+vw}$.

Now consider a pulse of light with wavelength measured at S to be λ_e and travelling in the direction of increasing x. Measured w.r.t. S, one cycle completes in Δt. The distance of the next wave front from O' is $\lambda_e + v\Delta t = \Delta t$. Then $\Delta t = \frac{\lambda_e}{1-v}$.

$$\begin{pmatrix} \Delta t \\ \Delta x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t' \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So,

$\lambda_o = \Delta t' = \frac{\Delta t}{cosh(\alpha)} = \frac{\lambda_e}{(1-v)cosh(\alpha)} = \lambda_e\sqrt{\frac{1+v}{1-v}}$

$= \lambda_e exp(tanh^{-1}v) = \lambda_e\sqrt{\frac{cosh(\alpha)+sinh(\alpha)}{cosh(\alpha)-sinh(\alpha)}}$ where λ_o is the observed wavelength. Consequently, $v = tanh(ln(z+1))$ where $z+1 = \frac{\lambda_o}{\lambda_e}$.

Now, consider a frame S' accelerating uniformly w.r.t. S.

Let $S \sim T, X, Y, Z$ be stationary coordinates and let $S' \sim t, x, y, z$ be the coordinates in an accelerated frame with relative acceleration $= \beta$ along the x direction for both systems. The Rindler coordinates are:

Let

$$\begin{aligned} T &= xsinh(\beta t) \\ X &= xcosh(\beta t) \end{aligned}$$

The inverse transformation is

$$
\begin{aligned}
t &= \frac{1}{\beta} tanh^{-1}\left(\frac{T}{X}\right) \\
x &= \sqrt{X^2 - T^2}
\end{aligned}
$$

Then $\left(\begin{array}{c} dT \\ dX \end{array} \right) = \left(\begin{array}{cc} \beta x cosh(\beta t) & sinh(\beta t) \\ \beta x sinh(\beta t) & cosh(\beta t) \end{array} \right) \left(\begin{array}{c} dt \\ dx \end{array} \right)$

and

$dS^2 = dT^2 - dX^2$

$= (\beta x cosh(\beta t)dt + sinh(\beta t)dx)^2 - (\beta x sinh(\beta t)dt + cosh(\beta t)dx)^2$

$= (\beta x)^2 dt^2 - dx^2$. For $x = constant$, $X^2 - T^2 = x^2$ forms an hyperbola in Minkowski space with x as the semi-major axis.

The world-lines in a Minkowski diagram consist of a foliation of hyperbolae indexed by x. The radial lines represent time. Where an hyperbola intersects a radial line corresponding to a given time value (t in the Rindler frame) tells us its position in the frame X, T. The speed of light varies along x in the accelerated frame, $v_{light} = \frac{|dx|}{|dt|} = \beta x$ where $0 < x < \infty$. This should not be seen as a violation of Special Relativity because in different coordinates (Kottler-Moller) the zero point of the accelerated frame is with a specific observer. In this case $v_{light} = 1 + \beta x$. For $x = 0$, $v_{light} = 1$.

Hyperbolic Expansion:

Define the pseudo-sphere in $\mathbf{R}^5$ as $R^2 = x^2 + y^2 + z^2 + w^2$ where

$$
\begin{aligned}
x &= R cosh\chi \\
y &= iR sinh\chi cos\theta \\
z &= iR sinh\chi sin\theta cos\phi \\
w &= iR sinh\chi sin\theta sin\phi
\end{aligned}
$$

Then

$$
\frac{\partial}{\partial R} = \frac{\partial x}{\partial R}\frac{\partial}{\partial x} + \frac{\partial y}{\partial R}\frac{\partial}{\partial y} + \frac{\partial z}{\partial R}\frac{\partial}{\partial z} + \frac{\partial w}{\partial R}\frac{\partial}{\partial w}
$$

$$\begin{aligned}
\frac{\partial}{\partial\chi} &= \frac{\partial x}{\partial\chi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\chi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\chi}\frac{\partial}{\partial z} + \frac{\partial w}{\partial\chi}\frac{\partial}{\partial w} \\
\frac{\partial}{\partial\theta} &= \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z} + \frac{\partial w}{\partial\theta}\frac{\partial}{\partial w} \\
\frac{\partial}{\partial\phi} &= \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z} + \frac{\partial w}{\partial\phi}\frac{\partial}{\partial w}
\end{aligned}$$

and

$$\begin{aligned}
h_R^2 &= \langle\frac{\partial}{\partial R},\frac{\partial}{\partial R}\rangle = (\frac{\partial x}{\partial R})^2 + (\frac{\partial y}{\partial R})^2 + (\frac{\partial z}{\partial R})^2 + (\frac{\partial w}{\partial R})^2 = 1 \\
h_\chi^2 &= \langle\frac{\partial}{\partial\chi},\frac{\partial}{\partial\chi}\rangle = (\frac{\partial x}{\partial\chi})^2 + (\frac{\partial y}{\partial\chi})^2 + (\frac{\partial z}{\partial\chi})^2 + (\frac{\partial w}{\partial\chi})^2 = -R^2 \\
h_\theta^2 &= \langle\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\rangle = (\frac{\partial x}{\partial\theta})^2 + (\frac{\partial y}{\partial\theta})^2 + (\frac{\partial z}{\partial\theta})^2 + (\frac{\partial w}{\partial\theta})^2 = -R^2 sinh^2\chi \\
h_\phi^2 &= \langle\frac{\partial}{\partial\phi},\frac{\partial}{\partial\phi}\rangle = (\frac{\partial x}{\partial\phi})^2 + (\frac{\partial y}{\partial\phi})^2 + (\frac{\partial z}{\partial\phi})^2 + (\frac{\partial w}{\partial\phi})^2 = -R^2 sinh^2\chi sin^2\theta
\end{aligned}$$

Then the metric on the 3-pseudo-sphere is

$$d\sigma^2 = R^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$$

Following Barrett O'Neill in *Semi-Riemannian Geometry* (1983), we derive the cosmological equations from GR:

Let M be a semi-Riemannian manifold, $\Psi(M)$ the space of differentiable vector fields on M, $g = \langle\cdot,\cdot\rangle$ a metric on M and D the Levi-Civita connection. The function $R : \Psi(M)^3 \to \Psi(M)$ given by

$$R_{XY}Z = D_{[X,Y]}Z - [D_X, D_Y]Z$$

is called the Riemannian curvature tensor on M.

The Ricci curvature tensor *Ric* of M is the contraction of R, in coordinates given by $R_{ij} = \Sigma_m R^m_{ijm}$ and the scalar curvature S is the contraction of *Ric*, in coordinates given by $S = \Sigma_{ij} g^{ij} R_{ij} = \Sigma_{ij} g^{ij} \Sigma_m R^m_{ijm}$.

The GR field equation is $Ric - \frac{1}{2}gS = 8\pi GT$ where

$T = (\rho + p)U^* \otimes U^* + pg$ where U^* is the tensor dual of U, the time directed flow vector in Robertson-Walker space-time orthogonal to a hypersurface of constant cosmic time.

Then $Ric - \frac{1}{2}gS = 8\pi G[(\rho + p)U^* \otimes U^* + pg]$

Then

$$\begin{aligned}
Ric(U,U) &= -3(\ddot{a}/a) \\
Ric(U,X) &= 0 \ for \ all \ X \perp U \\
Ric(X,Y) &= [2(\dot{a}/a)^2 + 2k/a^2 + \ddot{a}/a]g(X,Y) \ for \ all \ X,Y \perp U \\
S &= 6[(\dot{a}/a)^2 + k/a^2 + \ddot{a}/a]
\end{aligned}$$

where a is the expansion coefficient (radius for S^3), k is the tri-curvature, ρ is the mass-energy density, p is the pressure, and G is the gravitational constant.

$[Ric - \frac{1}{2}gS](X,Y) = Ric(X,Y) - \frac{1}{2}Sg(X,Y)$ which equals

$$Ric(U,U) - \frac{1}{2}Sg(U,U) + Ric(U,X) - \frac{1}{2}Sg(U,X) + Ric(X,Y) - \frac{1}{2}Sg(X,Y)$$

for all $X, Y \perp U$.

Now applying the field equation:

$$\begin{aligned}
Ric(U,U) - \frac{1}{2}Sg(U,U) &= 8\pi GT(U,U) \\
Ric(U,X) - \frac{1}{2}Sg(U,X) &= 8\pi GT(U,X) \ for \ all \ X \perp U \\
Ric(X,Y) - \frac{1}{2}Sg(X,Y) &= 8\pi GT(X,Y) \ for \ all \ X,Y \perp U
\end{aligned}$$

A metric signature (-,+,+,+) simplifies the calculation here so $g(U,U) = -1$,

$Ric(U,U) - \frac{1}{2}Sg(U,U) = 8\pi GT(U,U) = 8\pi G(\rho + p - p)$ so

$-3(\ddot{a}/a) - \frac{1}{2}6[(\dot{a}/a)^2 + k/a^2 + \ddot{a}/a]g(U,U)$

$= 3(\dot{a}/a)^2 + 3k/a^2 = 8\pi G\rho$

Then $3(\dot{a}/a)^2 + 3k/a^2 = 8\pi G\rho$

$Ric(U,X) - \frac{1}{2}S < U,X >= 8\pi GT(U,X) = 0$

for all $X, Y \perp U$.

Now, since $U^*(X) = U^*(Y) = 0$,

$Ric(X,Y) - \frac{1}{2}Sg(X,Y) = 8\pi GT(X,Y) = 8\pi Gpg(X,Y)$

for all $X, Y \perp U$.

$Ric(X,Y) - \frac{1}{2}Sg(X,Y) =$

$[2(\dot{a}/a)^2 + 2k/a^2 + \ddot{a}/a]g(X,Y) - \frac{1}{2}6[(\dot{a}/a)^2 + k/a^2 + \ddot{a}/a]g(X,Y)$

$= 8\pi Gpg(X,Y)$

$[2(\dot{a}/a)^2 + 2k/a^2 + \ddot{a}/a - 3[(\dot{a}/a)^2 + k/a^2 + \ddot{a}/a]]g(X,Y)$

$= 8\pi Gpg(X,Y)$

$[-(\dot{a}/a)^2 - k/a^2 - 2\ddot{a}/a]g(X,Y) = 8\pi Gpg(X,Y)$

for all $X, Y \perp U$.

Then

$-(\dot{a}/a)^2 - k/a^2 - 2\ddot{a}/a = 8\pi Gp$

So, the cosmological equations are:

$$\begin{aligned} \frac{8\pi G\rho}{3} &= \frac{k}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 && (1) \\ 8\pi Gp &= -2\frac{\ddot{a}}{a} - \frac{k}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 && (2) \\ \frac{8\pi G}{3}(\rho + 3p) &= -2\frac{\ddot{a}}{a} && (3) \end{aligned}$$

In equation (1) and (2), $k = -1, 0, 1$ corresponding to the tri-curvature of negative, flat and positive. The Hubble parameter $H = \frac{\dot{a}}{a} = \frac{1}{t_H}$. The units of the Hubble parameter are in s^{-1} allowing us to define Hubble time (now) as $t_H = H_0^{-1}$ so $H_0 = t_H^{-1}$ is true by definition.

So, the Robertson-Walker metric for the hyperbolic Universe is

$$ds^2 = dt^2 - a(t)^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$$

-the metric for the Universe with spatial description as S_P^3. In a hyperbolic Universe $a = \frac{A}{2}(cosh\eta - 1)$ where A is a constant and η is conformal time.

The pseudo-sphere S_p^3 can be defined multiple ways. Here we are interested in defining it similarly to that above as a pseudo-sphere in $\mathbf{R}^5$ defined by $R^2 = x^2 + y^2 + z^2 + w^2$ where

$$\begin{aligned}
x &= \mathbf{1}Rcosh\chi \\
y &= \mathbf{i}Rsinh\chi cos\theta \\
z &= \mathbf{j}Rsinh\chi sin\theta cos\phi \\
w &= \mathbf{k}Rsinh\chi sin\theta sin\phi
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial R} &= \frac{\partial x}{\partial R}\frac{\partial}{\partial x} + \frac{\partial y}{\partial R}\frac{\partial}{\partial y} + \frac{\partial z}{\partial R}\frac{\partial}{\partial z} + \frac{\partial w}{\partial R}\frac{\partial}{\partial w} \\
\frac{\partial}{\partial \chi} &= \frac{\partial x}{\partial \chi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \chi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \chi}\frac{\partial}{\partial z} + \frac{\partial w}{\partial \chi}\frac{\partial}{\partial w} \\
\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} + \frac{\partial w}{\partial \theta}\frac{\partial}{\partial w} \\
\frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} + \frac{\partial w}{\partial \phi}\frac{\partial}{\partial w}
\end{aligned}$$

where

$$\begin{pmatrix}
\frac{\partial x}{\partial R} & \frac{\partial y}{\partial R} & \frac{\partial z}{\partial R} & \frac{\partial w}{\partial R} \\
\frac{\partial x}{\partial \chi} & \frac{\partial y}{\partial \chi} & \frac{\partial z}{\partial \chi} & \frac{\partial w}{\partial \chi} \\
\frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} & \frac{\partial w}{\partial \theta} \\
\frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} & \frac{\partial w}{\partial \phi}
\end{pmatrix}$$

$$
= \begin{pmatrix} \mathbf{1}cosh\chi & \mathbf{i}sinh\chi cos\theta & \mathbf{j}sinh\chi sin\theta cos\phi & \mathbf{k}sinh\chi sin\theta sin\phi \\ \mathbf{1}Rsinh\chi & \mathbf{i}Rcosh\chi cos\theta & \mathbf{j}Rcosh\chi sin\theta cos\phi & \mathbf{k}Rcosh\chi sin\theta sin\phi \\ 0 & -\mathbf{i}Rsinh\chi sin\theta & \mathbf{j}Rsinh\chi cos\theta cos\phi & \mathbf{k}Rsinh\chi cos\theta sin\phi \\ 0 & 0 & -\mathbf{j}Rsinh\chi sin\theta sin\phi & \mathbf{k}Rsinh\chi sin\theta cos\phi \end{pmatrix}
$$

Then

$$
\begin{aligned}
h_R^2 &= \langle\frac{\partial}{\partial R}, \frac{\partial}{\partial R}\rangle = (\frac{\partial x}{\partial R})^2 + (\frac{\partial y}{\partial R})^2 + (\frac{\partial z}{\partial R})^2 + (\frac{\partial w}{\partial R})^2 = \mathbf{1} \\
h_\chi^2 &= \langle\frac{\partial}{\partial \chi}, \frac{\partial}{\partial \chi}\rangle = (\frac{\partial x}{\partial \chi})^2 + (\frac{\partial y}{\partial \chi})^2 + (\frac{\partial z}{\partial \chi})^2 + (\frac{\partial w}{\partial \chi})^2 = -\mathbf{1}R^2 \\
h_\theta^2 &= \langle\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\rangle = (\frac{\partial x}{\partial \theta})^2 + (\frac{\partial y}{\partial \theta})^2 + (\frac{\partial z}{\partial \theta})^2 + (\frac{\partial w}{\partial \theta})^2 = -\mathbf{1}R^2 sinh^2\chi \\
h_\phi^2 &= \langle\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}\rangle = (\frac{\partial x}{\partial \phi})^2 + (\frac{\partial y}{\partial \phi})^2 + (\frac{\partial z}{\partial \phi})^2 + (\frac{\partial w}{\partial \phi})^2 = -\mathbf{1}R^2 sinh^2\chi sin^2\theta
\end{aligned}
$$

The metric $ds^2 = dR^2 - R^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$ describes a four dimensional flat space. It is similar to the Robertson-Walker metric $ds^2 = dt^2 - R^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$ which describes a curved four dimensional space-time. The difference is dR is replaced by dt in the first expression.

If, however, we make $Rcosh\chi$ a proxy for time and we set

$$
\begin{aligned}
T &= Rcosh\chi && (4) \\
X &= Rsinh\chi cos\theta && (5) \\
Y &= Rsinh\chi sin\theta cos\phi && (6) \\
Z &= Rsinh\chi sin\theta sin\phi && (7)
\end{aligned}
$$

Then $R^2 = T^2 - X^2 - Y^2 - Z^2$ is a flat space-time invariant with respect to spatial rotations in θ and ϕ. It is a flattened out version of four dimensional hyperbolic space-time.

It has the usual Lorentz invariance properties.

The above quadratic written as

$$\begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix}$$

is invariant under every Lorentz transformation Λ

$$\begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix}^T \Lambda^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} T \\ X \\ Y \\ Z \end{pmatrix}$$

with R^2 as an invariant. Using the usual method we can deduce that

$$dT^2 - dX^2 - dY^2 - dZ^2 = dR^2 - R^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\theta d\phi^2)]$$

Setting $r = Rsinh\chi$, equations (5-7) are ordinary spherical coordinates in flat space. The *sinh* function grows nearly exponentially causing $r = Rsinh\chi$ to grow nearly exponentially but it is not R doing so but the $sinh\chi$ factor. This results in the distance to objects being further away that expected in a flat space model. Not only are they further away than expected but they also appear further away than they actually are due to the divergence in lines of flux resulting from negative curvature of space.

We can represent the three optical distances* as follows (Carroll, Sean, 2019, p.348) and (Peebles, PJE, 1993, p.319):

For $k = +1$:

$$D_A = (1+z)^{-1} H_0^{-1} |\Omega_{k=+1}|^{-1/2} sin\left[|\Omega_{k=+1}|^{1/2} \int_0^z \frac{dz'}{E(z')}\right]$$

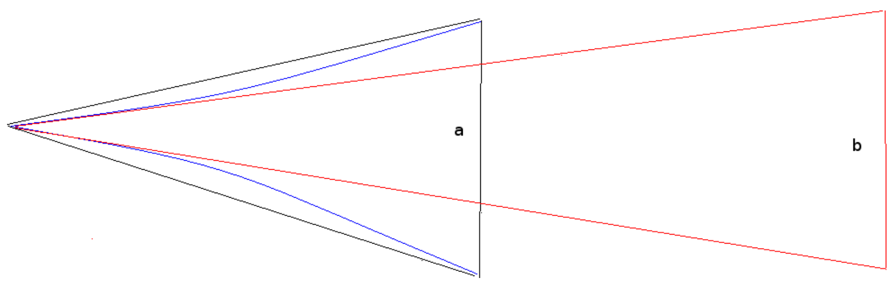
For $k = 0$:

$$D_A = (1+z)^{-1} H_0^{-1} \int_0^z \frac{dz'}{E(z')}$$

For $k = -1$:

$$D_A = (1+z)^{-1} H_0^{-1} |\Omega_{k=-1}|^{-1/2} sinh\left[|\Omega_{k=-1}|^{1/2} \int_0^z \frac{dz'}{E(z')} \right]$$

Consider two points of equal co-moving distance with one on the concave curve and the other on the straight line. As light passes through negatively curved space, parallel rays will diverge. So under the assumption of the inverse square law, the optical distance using a standard candle (e.g. a super nova of known brightness) will appear greater (for the same co-moving distance) than if space were flat. Consequently, in a negatively curved space, the co-moving distance of galaxies is greater (for the same look-back time) than in flat space and they also appear even more distant due to the reduction in electromagnetic flux.



In the above figure the black lines represent rays traveling through flat space from **a** and converging on a focal point. The blue lines represent rays traveling from **a** through negatively curved space. The red triangle represents how the object appears further and dimmer at **b**.

In the article *Quaternion Space-time (Part 1)* we saw that

$$exp(\mathbf{H}) = \mathbf{R}^{+} \times S^3$$

so we would like to compute $exp(i\mathbf{H}) = exp(i\tau + i\chi\mathbf{i} + i\theta\mathbf{j} + i\phi\mathbf{k})$.

Following along with the derivation in the above article

The pattern is:

$(i\chi\mathbf{i} + i\theta\mathbf{j} + i\phi\mathbf{k})^{2n} = (-1)^{2n}(\chi^2 + \theta^2 + \phi^2)^n = (\chi^2 + \theta^2 + \phi^2)^n$

$(i\chi\mathbf{i} + i\theta\mathbf{j} + i\phi\mathbf{k})^{2n+1}$

$= i(-1)^n[\chi(\chi^2+\theta^2+\phi^2)^n\mathbf{i}+\theta(\chi^2+\theta^2+\phi^2)^n\mathbf{j}+\phi(\chi^2+\theta^2+\phi^2)^n\mathbf{k}]$

$exp(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k}) = \Sigma_{n=0}^{\infty}\frac{(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k})^n}{n!}$

$= \Sigma_{n=0}^{\infty}\frac{(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k})^{2n}}{(2n)!} + \Sigma_{n=0}^{\infty}\frac{(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k})^{2n+1}}{(2n+1)!}$

$= \Sigma_{n=0}^{\infty}\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n)!}$

$+i\chi\Sigma_{n=0}^{\infty}\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}\mathbf{i}+i\theta\Sigma_{n=0}^{\infty}\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}\mathbf{j}+i\phi\Sigma_{n=0}^{\infty}\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}\mathbf{k}$

Let $\alpha = \Sigma_{n=0}^{\infty}\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n)!}$ and $\beta = \Sigma_{n=0}^{\infty}\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}$

Then $exp\begin{pmatrix} 0 & -i\chi & i\theta & -i\phi \\ i\chi & 0 & i\phi & i\theta \\ -i\theta & -i\phi & 0 & i\chi \\ i\phi & -i\theta & -i\chi & 0 \end{pmatrix} = \begin{pmatrix} \alpha & -i\chi\beta & i\theta\beta & -i\phi\beta \\ i\chi\beta & \alpha & i\phi\beta & i\theta\beta \\ -i\theta\beta & -i\phi\beta & \alpha & i\chi\beta \\ i\phi\beta & -i\theta\beta & -i\chi\beta & \alpha \end{pmatrix}$

$= \alpha I + \beta\begin{pmatrix} 0 & -i\chi & i\theta & -i\phi \\ i\chi & 0 & i\phi & i\theta \\ -i\theta & -i\phi & 0 & i\chi \\ i\phi & -i\theta & -i\chi & 0 \end{pmatrix} = \alpha\mathbf{1}+\beta(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k})$

For the case $\chi \neq 0, \theta = 0, \phi = 0$

$\alpha = \Sigma_{n=0}^{\infty}\frac{(\chi^2)^n}{(2n)!} = \Sigma_{n=0}^{\infty}\frac{\chi^{2n}}{(2n)!} = cosh\chi$

and $i\chi\beta = i\chi\Sigma_{n=0}^{\infty}\frac{(\chi^2)^n}{(2n+1)!} = i\Sigma_{n=0}^{\infty}\frac{\chi^{2n+1}}{(2n+1)!} = isinh\chi$

Then $exp\begin{pmatrix} 0 & -i\chi & 0 & 0 \\ i\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & i\chi \\ 0 & 0 & -i\chi & 0 \end{pmatrix} = \begin{pmatrix} cosh\chi & -isinh\chi & 0 & 0 \\ isinh\chi & cosh\chi & 0 & 0 \\ 0 & 0 & cosh\chi & isinh\chi \\ 0 & 0 & -isinh\chi & cosh\chi \end{pmatrix}$

$= cosh\chi\mathbf{1}+isinh\chi\mathbf{i}$

For a quaternion $\mathbf{q} = a\mathbf{1}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$, $\|\mathbf{q}\| = \sqrt{\mathbf{q}\mathbf{q}^*}$

where $\mathbf{q}^* = a\mathbf{1}-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}$

Then $\|cosh\chi\mathbf{1}+isinh\chi\mathbf{i}\| = \sqrt{cosh^2\chi - sinh^2\chi} = 1$

For the case $\chi = 0, \theta \neq 0, \phi = 0$

$\alpha = \Sigma_{n=0}^{\infty} \frac{(\theta^2)^n}{(2n)!} = \Sigma_{n=0}^{\infty} \frac{\theta^{2n}}{(2n)!} = cosh\theta$

and $i\theta\beta = i\theta\Sigma_{n=0}^{\infty} \frac{(\theta^2)^n}{(2n+1)!} = i\Sigma_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!} = isinh\theta$

$$\text{Then } exp\begin{pmatrix} 0 & 0 & i\theta & 0 \\ 0 & 0 & 0 & i\theta \\ -i\theta & 0 & 0 & 0 \\ 0 & -i\theta & 0 & 0 \end{pmatrix} = \begin{pmatrix} cosh\theta & 0 & isinh\theta & 0 \\ 0 & cosh\theta & 0 & isinh\theta \\ -isinh\theta & 0 & cosh\theta & 0 \\ 0 & -isinh\theta & 0 & cosh\theta \end{pmatrix}$$

$= cosh\theta\mathbf{1} + isinh\theta\mathbf{j}$

$\|cosh\theta\mathbf{1} + isinh\theta\mathbf{j}\| = \sqrt{cosh^2\theta - sinh^2\theta} = 1$

For the case $\chi = 0, \theta = 0, \phi \neq 0$

$\alpha = \Sigma_{n=0}^{\infty} \frac{(\phi^2)^n}{(2n)!} = \Sigma_{n=0}^{\infty} \frac{\phi^{2n}}{(2n)!} = cosh\phi$

and $i\phi\beta = i\phi\Sigma_{n=0}^{\infty} \frac{(\phi^2)^n}{(2n+1)!} = i\Sigma_{n=0}^{\infty} \frac{\phi^{2n+1}}{(2n+1)!} = isinh\phi$

$$exp\begin{pmatrix} 0 & 0 & 0 & -i\phi \\ 0 & 0 & i\phi & 0 \\ 0 & -i\phi & 0 & 0 \\ i\phi & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} cosh\phi & 0 & 0 & -isinh\phi \\ 0 & cosh\phi & isinh\phi & 0 \\ 0 & -isinh\phi & cosh\phi & 0 \\ isinh\phi & 0 & 0 & cosh\phi \end{pmatrix}$$

$= cosh\phi\mathbf{1} + isinh\phi\mathbf{k}$

$\|cosh\phi\mathbf{1} + isinh\phi\mathbf{k}\| = \sqrt{cosh^2\phi - sinh^2\phi} = 1$

Furthermore,

$$[\alpha\mathbf{1}+\beta(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k})][\alpha\mathbf{1}-\beta(i\chi\mathbf{i}+i\theta\mathbf{j}+i\phi\mathbf{k})] = \mathbf{1}(\alpha^2-\beta^2\chi^2-\beta^2\theta^2-\beta^2\phi^2)$$

$$\text{Then } exp(i\mathbf{H}) = \{e^{i\tau} exp\begin{pmatrix} 0 & -i\chi & i\theta & -i\phi \\ i\chi & 0 & i\phi & i\theta \\ -i\theta & -i\phi & 0 & i\chi \\ i\phi & -i\theta & -i\chi & 0 \end{pmatrix} : (\tau, \chi, \theta, \phi) \in \mathbf{R}^4\}$$

$\sim S^1 \times S^3_P$ since $\{\mathbf{q} : ||\mathbf{q}|| = \sqrt{\alpha^2 - \beta^2\chi^2 - \beta^2\theta^2 - \beta^2\phi^2} = 1\} \cong S^3_P$.

So, $exp(i\mathbf{H}) \sim S^1 \times S_P^3$. Then changing $i\tau$ to τ

$$\{e^{\tau} exp \begin{pmatrix} 0 & -i\chi & i\theta & -i\phi \\ i\chi & 0 & i\phi & i\theta \\ -i\theta & -i\phi & 0 & i\chi \\ i\phi & -i\theta & -i\chi & 0 \end{pmatrix} : (\tau, \chi, \theta, \phi) \in \mathbf{R}^4\} = \mathbf{R}^+ \times S_P^3$$

Now, consider the exponential of

$$\mathbf{S} = \{\tau\sigma_0 + x\sigma_x + y\sigma_y + z\sigma_z : \tau, x, y, z \in \mathbf{R}\}$$

$\mathbf{S}$ is the span of the Pauli matrices including the identity matrix $\sigma_0 = I$.

σ_0 commutes with the other Pauli matrices and there is an isomorphism $(\sigma_x, \sigma_y, \sigma_z) \leftrightarrow (\mathbf{i}, \mathbf{j}, \mathbf{k})$ given by $\sigma_x \leftrightarrow i\mathbf{i}$, $\sigma_y \leftrightarrow i\mathbf{j}$, $\sigma_z \leftrightarrow i\mathbf{k}$.

Consequently, $e^{\mathbf{S}} = \mathbf{R}^+ \times S_P^3$ which we assert is the topology of the Universe.

As mentioned above, we can show $e^{\mathbf{H}} = \mathbf{R}^+ \times S^3$ where $\mathbf{H} = span\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, the span (over $\mathbf{R}$) of quaternions.

For $\mathbf{R} \times \mathbf{R}^3 = \mathbf{R}^4$, there is no space $\mathbf{X}$ s.t. $e^{\mathbf{X}} = \mathbf{R}^+ \times \mathbf{R}^3$ but the Lie algebra (the space of infinitesimal generators) is $\mathbf{R}^4$.

The Universe has the three possible topologies:

Flat: $\mathbf{R} \times \mathbf{R}^3$. The Lie algebra is $\mathbf{R}^4$

Positively curved: $\mathbf{R}^+ \times S^3$ where S^3 is the 3-sphere. Its tangent space is $\mathbf{H}$, the span of quaternions.

Negatively curved: $\mathbf{R}^+ \times S_P^3$ where S_P^3 is the 3-pseudosphere. Its tangent space is $\mathbf{S}$, the span of Pauli matrices which has relevance for the existence of fermions.

1/2-Spin Rotations

The quaternions are associated with spatial rotations of the form $\mathbf{v}' = R_{(\mathbf{u},\theta)}(\mathbf{v}) = \mathbf{w}\mathbf{v}\mathbf{w}^{-1}$ where $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is an initial vector before rotation, $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ is a unit vector along the axis of rotation (Euler axis), θ is an angle of rotation, and

$\mathbf{w} = exp[\frac{\theta}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})] = cos\frac{\theta}{2}\mathbf{1} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})sin\frac{\theta}{2}$ and

$\mathbf{w}^{-1} = exp[-\frac{\theta}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})] = cos\frac{\theta}{2}\mathbf{1} - (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})sin\frac{\theta}{2}$

We readily observe that $R_{(\mathbf{u},2\pi)}(\mathbf{v}) = (-\mathbf{1})\mathbf{v}(-\mathbf{1}) = \mathbf{v}$ and $R_{(\mathbf{u},4\pi)}(\mathbf{v}) = (\mathbf{1})\mathbf{v}(\mathbf{1}) = \mathbf{v}$.

We can define the 2-to-1 surjective homomorphism $\phi : \mathbf{H} \to SO(3)\backslash\{-I\}$

by $\phi(A) = \phi(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})$

$$= -\frac{1}{2}\begin{pmatrix} Tr(\mathbf{i}A\mathbf{i}A^{-1}) & Tr(\mathbf{i}A\mathbf{j}A^{-1}) & Tr(\mathbf{i}A\mathbf{k}A^{-1}) \\ Tr(\mathbf{j}A\mathbf{i}A^{-1}) & Tr(\mathbf{j}A\mathbf{j}A^{-1}) & Tr(\mathbf{j}A\mathbf{k}A^{-1}) \\ Tr(\mathbf{k}A\mathbf{i}A^{-1}) & Tr(\mathbf{k}A\mathbf{j}A^{-1}) & Tr(\mathbf{k}A\mathbf{k}A^{-1}) \end{pmatrix}$$

$$= -\begin{pmatrix} (c^2+d^2)-(a^2+b^2) & 2(ad-bc) & -2(ac+bd) \\ -2(ad+bc) & (b^2+d^2)-(a^2+c^2) & 2(ab-cd) \\ 2(ac-bd) & -2(cd+ab) & (c^2+b^2)-(a^2+d^2) \end{pmatrix}$$

where $Tr(q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) = q_0$.

It is evident that $\phi(A) = \phi(-A)$ so the mapping is 2-to-1 and $\phi(\mathbf{1}) = \phi(-\mathbf{1}) = I$ so $Ker(\phi) = \{\mathbf{1}, -\mathbf{1}\}$.*

The rotation θ must be with respect to the Euler axis so $R_{(\mathbf{u},\theta)}(\mathbf{v}) = R_{(-\mathbf{u},-\theta)}\mathbf{v}$ are equal rotations in SO(3) corresponding to $\phi(A) = \phi(-A)$ where A and $-A$ have reversed parity. In $\mathbf{H}$ the mapping $A \to -A$ corresponds to a $1/2$ rotation+(multiple full rotations). That is, $-A = e^{i(\pi+n2\pi)}A$. Compare this to spinor rotation as discussed in *Pauli and Dirac Matrices* where the rotation operator on a 4×2 spinor is

$$R(\theta)\begin{pmatrix} g & -h \\ h & g \\ k & -l \\ l & k \end{pmatrix} = (cos\frac{\theta}{2}\mathbf{1} + sin\frac{\theta}{2}(n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}))\begin{pmatrix} g & -h \\ h & g \\ k & -l \\ l & k \end{pmatrix}$$

$$= \begin{pmatrix} cos(\theta/2) & n_z sin(\theta/2) & -n_y sin(\theta/2) & n_x sin(\theta/2) \\ -n_z sin(\theta/2) & cos(\theta/2) & -n_x sin(\theta/2) & -n_y sin(\theta/2) \\ n_y sin(\theta/2) & n_x sin(\theta/2) & cos(\theta/2) & -n_z sin(\theta/2) \\ -n_x sin(\theta/2) & n_y sin(\theta/2) & n_z sin(\theta/2) & cos(\theta/2) \end{pmatrix} \begin{pmatrix} g & -h \\ h & g \\ k & -l \\ l & k \end{pmatrix}$$

where $\mathbf{n}$ is the unit Euler axis corresponding to the unit rotation axis $\mathbf{u}$ referred to above. For such a rotation, $R(\theta) \neq R(\theta + n2\pi)$ (n odd) but $R(\theta) = R(\theta + n2\pi)$ (n even). Relative to the rotation group $SO(3)$ there is no difference between $\phi(A)$ and $\phi(-A)$ though relative to $\mathbf{H}$ there is a 1/2 rotation $A \to -A$. Applying $R(\theta)$ to a general quaternion $\mathbf{q} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ gives

$R(\theta)(\mathbf{q}) = (acos\frac{\theta}{2} + sin\frac{\theta}{2}(-bn_x - cn_y - dn_z))\mathbf{1} + (bcos\frac{\theta}{2} + sin\frac{\theta}{2}(an_x + dn_y - cn_z))\mathbf{i} + (ccos\frac{\theta}{2} + sin\frac{\theta}{2}(-dn_x + an_y + bn_z))\mathbf{j} + (dcos\frac{\theta}{2} + sin\frac{\theta}{2}(cn_x - bn_y + an_z))\mathbf{k}$

We note that $R(2\pi)(\mathbf{q}) = -\mathbf{q}$ but $R(4\pi)(\mathbf{q}) = \mathbf{q}$. So, like a Möbius Strip, one time around reverses the orientation but twice around restores it.

Now, a rotation in Pauli spin space is given by $\hat{U}|\alpha\rangle$ where

$$\hat{U} = exp(-\frac{i}{\hbar}\theta\mathbf{u}\cdot\hat{\mathbf{S}}) = \begin{pmatrix} cos(\frac{\theta}{2}) & -sin(\frac{\theta}{2})e^{-i\phi} \\ sin(\frac{\theta}{2})e^{i\phi} & cos(\frac{\theta}{2}) \end{pmatrix}$$

With the simplifyinjg assumption that the rotation is around the z axis we then have $\phi = 0$ and $e^{i\phi} = 1$ and

$$\hat{U} = \begin{pmatrix} cos(\frac{\theta}{2}) & -sin(\frac{\theta}{2}) \\ sin(\frac{\theta}{2}) & cos(\frac{\theta}{2}) \end{pmatrix}$$

Then with a $\theta = 2\pi$ rotation

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

which reverses the orientation. Applying the operator $\hat{U}$ again restores the original orientation. Think of the Mobius strip. One time around reverses orientation. Twice around restores it.

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