The Emissions of Matter and Radiation Near the Axis of Rotation of a Rapidly Rotating Kerr Black Hole

Abstract: We analyze the Kerr metric using a method of foliations. We use the original mathematical methods of Karl Schwarzschild where he defined the area radius as $R = (r^3 + \alpha^3)^{1/3}$. This makes a considerable difference and allows us to explain the emission along the axis of rotation of both matter and radiation.

The same analysis as in *The Schwarzschild Metric* can be carried out for rotating gravitating bodies using the Kerr metric which is the axially symmetric solution around a rotating gravitating body with angular momentum J:

$$ds^2 = (1 - \alpha/\tilde{R})dt^2 + \frac{2\alpha a sin^2\theta}{\tilde{R}}dtd\phi - \frac{\Sigma}{\Delta}dR^2 - d\Omega^2$$

where c is set at 1, t is the elapsed time of a clock 'at infinity', r is the scalar distance, $R = (r^3 + \alpha^3)^{1/3}$, $\alpha = 2GM$, $\tilde{R} = \frac{R^2+a^2cos^2\theta}{R}$, $a = J/M$, $\Sigma = R^2 + a^2cos^2\theta$, $\Delta = R^2 - \alpha R + a^2$, and

$$d\Omega^2 = \Sigma d\theta^2 + (R^2 + a^2 + \frac{\alpha a^2 sin^2\theta}{\tilde{R}})sin^2\theta d\phi^2$$

Considering space as a foliation of 3D manifolds of revolution given by

$$\begin{aligned} u &= \tilde{R}cos\phi = \frac{R^2 + a^2cos^2\theta}{R}cos\phi \\ v &= \tilde{R}sin\phi = \frac{R^2 + a^2cos^2\theta}{R}sin\phi \\ w &= r \end{aligned}$$

where $r > 0$, $0 < \theta < \pi$ and $0 \leq \phi < 2\pi$. The angle θ is the co-latitude and ϕ is the longitudinal angle w.r.t. the axis of rotation (direction of increase according to the right hand rule relative to $\mathbf{a}$). Then $a = \pm||\mathbf{a}||$.

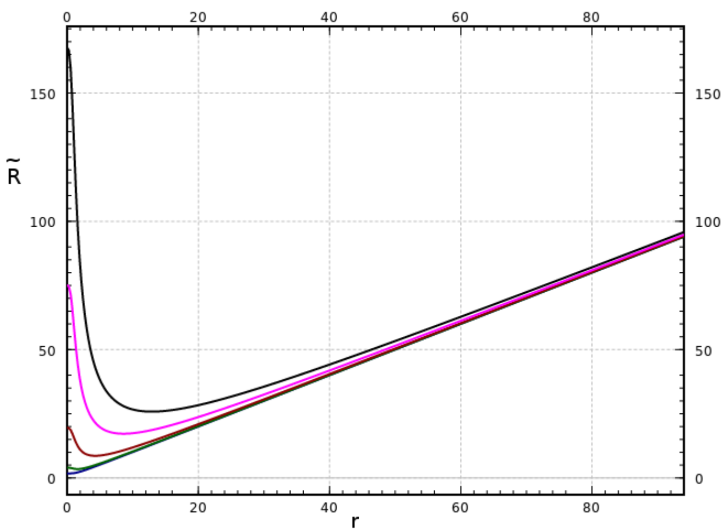
Setting $\alpha = 1$, each manifold in the foliation is a surface of revolution of $((r^3+1)^{2/3} + a^2cos^2\theta)/(r^3+1)^{1/3}$ about the r-axis (with θ constant).

The following table gives the values of $acos\theta$ for

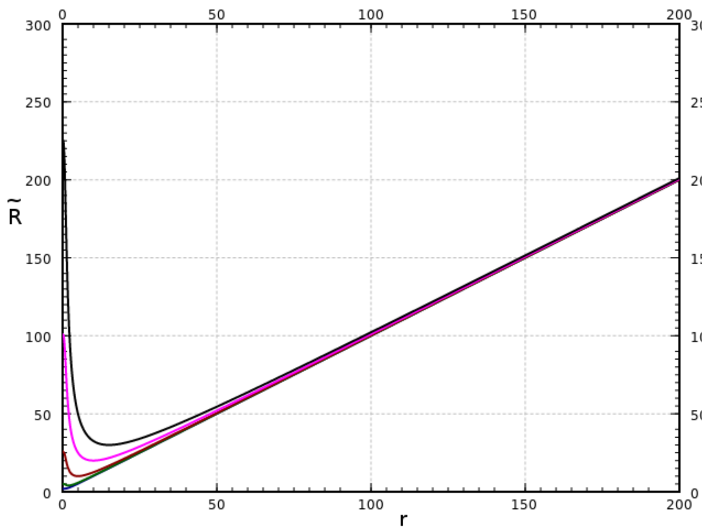
$a \in \{2/\sqrt{3}, 4/\sqrt{3}, 10/\sqrt{3}, 20/\sqrt{3}, 30/\sqrt{3}\}$:

$sin\theta$	$cos\theta$	$acos\theta$	$a^2cos^2\theta$
$2/3$	$\sqrt{5}/3$	$\frac{2\sqrt{5}}{3\sqrt{3}}, \frac{4\sqrt{5}}{3\sqrt{3}}, \frac{10\sqrt{5}}{3\sqrt{3}}, \frac{20\sqrt{5}}{3\sqrt{3}}, \frac{30\sqrt{5}}{3\sqrt{3}}$	$.741, 2.963, 18.519, 74.074, 166.667$
$1/2$	$\sqrt{3}/2$	$1, 2, 5, 10, 15$	$1, 4, 25, 100, 225$
$1/3$	$2\sqrt{2}/3$	$\frac{4\sqrt{2}}{3\sqrt{3}}, \frac{8\sqrt{2}}{3\sqrt{3}}, \frac{20\sqrt{2}}{3\sqrt{3}}, \frac{40\sqrt{2}}{3\sqrt{3}}, \frac{20\sqrt{2}}{\sqrt{3}}$	$1.33, 4.74, 29.63, 118.52, 266.67$
$1/10$	$\frac{\sqrt{99}}{10}$	$\frac{\sqrt{33}}{10}, \frac{\sqrt{33}}{5}, \frac{\sqrt{33}}{2}, \sqrt{33}, \frac{3\sqrt{33}}{2}$	$.337, 1.32, 8.25, 33.00, 74.25$

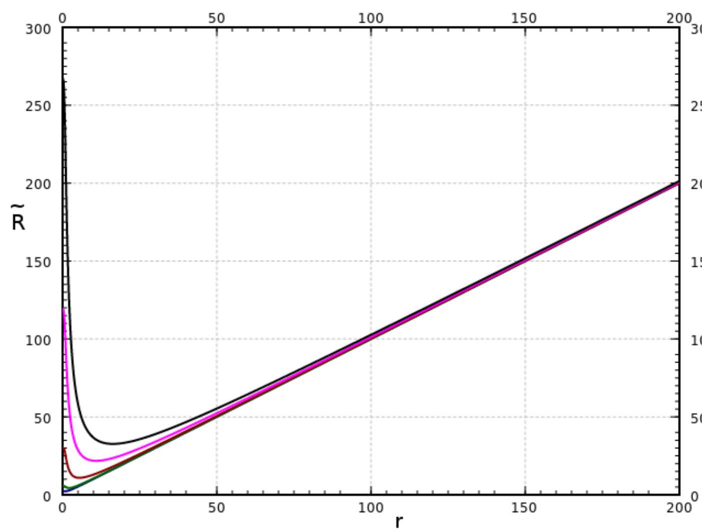
This gives us the following figures of $\tilde{R}$ v. r:



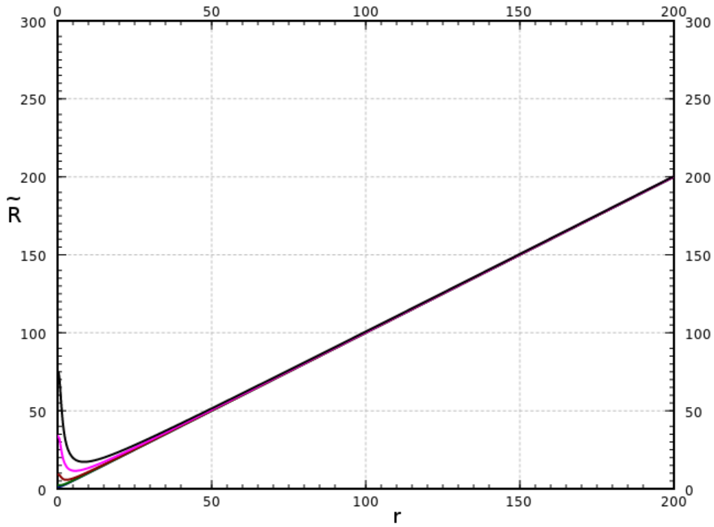
$sin\theta = 2/3$



$sin\theta = 1/2$



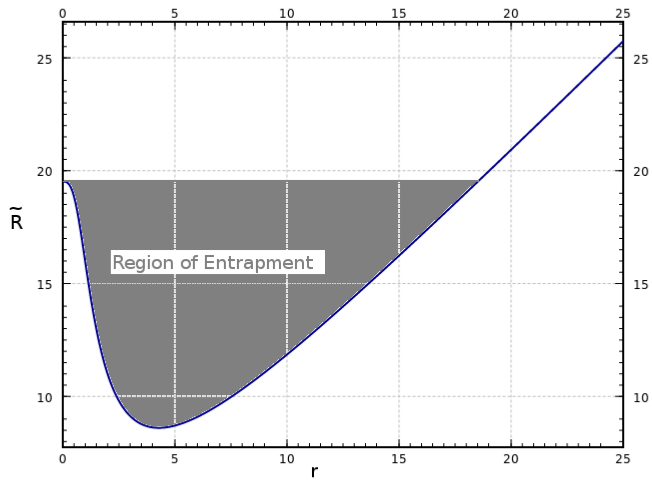
$sin\theta = 1/3$



$sin\theta = 1/10$

Particle trajectories can be trapped between the two values of r where the gravitational potentials $-\frac{GM}{\tilde{R}}$ are equal and can process around with constant ϕ angular momentum.

$R(0) = \alpha$ and $\tilde{R}(0) = \frac{\alpha^2+a^2cos^2\theta}{\alpha}$. Furthermore, $\tilde{R}'(R) = \frac{R^2-a^2cos^2\theta}{R^2}$ which equals zero when $R = |a|cos\theta$. $\tilde{R}''(R) = \frac{2a^2cos^2\theta}{R^3} > 0$ so $(R, \tilde{R}) = (|a|cos\theta, 2|a|cos\theta)$ is a minimum. $R \geq \alpha$ so $|a|cos\theta \geq \alpha$. If $R > \alpha$ then $|a|cos\theta > \alpha$. Then $cos\theta > \frac{\alpha}{|a|}$ ensures there is a region of entrapment of trajectories. Otherwise there is no entrapment of trajectories. Notice that the trapping surfaces are repelling because the potential is effective in the direction of decreasing $\tilde{R}$.

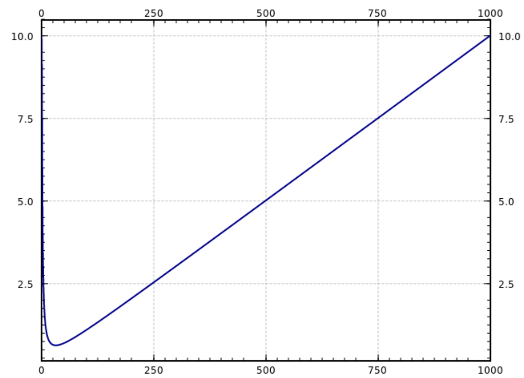


We can see in the following graphic an artist's rendition of what we are calling the entrapment region containing dust and gas swirling around as if caught in a giant tornado. As the material swirls around it emits EMR at various wavelengths. For material entering the entrapment region, θ is decreasing resulting in a boost to $d\phi/d\tau$ implied by the cross term $\frac{2\alpha a sin^2\theta}{\tilde{R}}\frac{dt}{d\tau}\frac{d\phi}{d\tau}$. The direction of the boost depends on the sign of a (since we measure the direction of ϕ by the right hand rule relative to $\mathbf{a}$). So, material is swirling clockwise relative to one pole and counter-clockwise relative to the other.

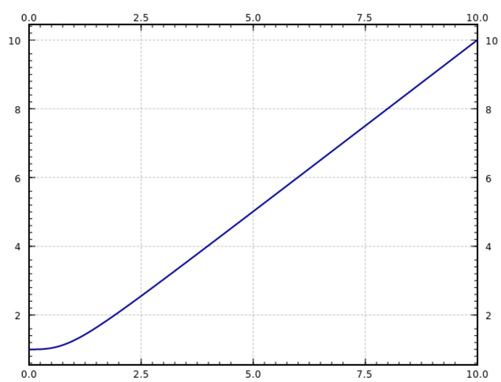
Image credit:NASA/JPL-CalTech

At $cos\theta = \frac{\alpha}{|a|}$ the entrapment breaks down and trajectories are free to leave. The accretion disk of a gravitating body is mostly in the equatorial plane where $\theta \approx \pi/2$ and would tend to satisfy $cos\theta < \frac{\alpha}{a}$ and not get into the entrapment region but fall into the center as in the non-rotating case. Particles which fall into the entrapment region will oscillate between two values of r depending on a given particle's energy. For $R > |a|cos\theta$ the same conclusions apply as for the non-rotating case. Light can escape but becomes extremely red shifted.

Though it is not evident from the small scale of the above graph, $\frac{\tilde{R}}{r} \to 1$ which it should so that $(1 - \frac{2GM}{\tilde{R}}) \approx (1 - \frac{2GM}{r})$ for large r. (Following figure)



For $cos\theta \leq \frac{\alpha}{|a|}$, the hyperboloid resembles that for the non-rotating case.



Suppose α grows without a corresponding growth is $|a|$. Then eventually for a given θ, $cos\theta \leq \frac{\alpha}{|a|}$ and entrapment would break down for that angle. This could happen with an abundance of material falling in along radial lines, contributing to the mass but not the angular momentum. Then particles oscillating at angle θ would suddenly be free to leave.

The Symmetric Bilinear Form for the Kerr Metric

The Kerr metric can be written in bilinear form as $\begin{pmatrix} dt \\ dR \\ d\theta \\ d\phi \end{pmatrix}^T K \begin{pmatrix} dt \\ dR \\ d\theta \\ d\phi \end{pmatrix}$

where $K = \begin{pmatrix} (1-\frac{\alpha}{\tilde{R}}) & 0 & 0 & \frac{\alpha a sin^2\theta}{\tilde{R}} \\ 0 & -\frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & -\Sigma & 0 \\ \frac{\alpha a sin^2\theta}{\tilde{R}} & 0 & 0 & -(R^2+a^2+\frac{\alpha a^2}{\tilde{R}} sin^2\theta) sin^2\theta \end{pmatrix}$

which we can write as $K = \begin{pmatrix} A & 0 & 0 & E \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ E & 0 & 0 & D \end{pmatrix}$

The eigenvalues for K are $\lambda = \frac{(A+D) \pm \sqrt{(A-D)^2+4E^2}}{2}$,$B$,$C$.

Then the diagonal bilinear form for the Kerr metric is

$$Q^T K Q = \begin{pmatrix} \frac{(A+D)+\sqrt{(A-D)^2+4E^2}}{2} & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & \frac{(A+D)-\sqrt{(A-D)^2+4E^2}}{2} \end{pmatrix}$$

where Q is orthogonal since K is real symmetric

and where $\begin{pmatrix} dt \\ dR \\ d\theta \\ d\phi \end{pmatrix} = Q \begin{pmatrix} dt' \\ dR' \\ d\theta' \\ d\phi' \end{pmatrix}$ is the orthogonal coordinate transformation between the two sets of coordinates. We say that a 4x4 symmetric bilinear form with real entries preserves *quaternion structure* if it has four real eigenvalues where one is positive and the others negative or zero (see the article *Quaternion Space-time*).

$B < 0$ and $C < 0$ so quaternion structure is preserved if

$$\frac{(A+D)+\sqrt{(A-D)^2+4E^2}}{2} > 0$$

and

$$\frac{(A+D)-\sqrt{(A-D)^2+4E^2}}{2} < 0$$

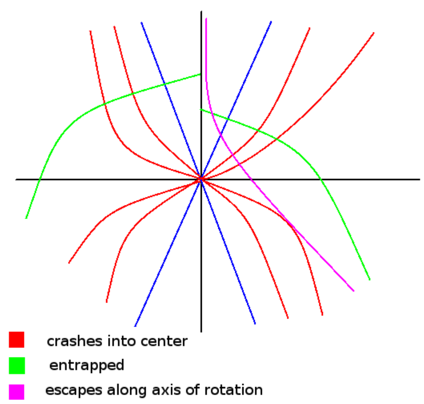
.

If either inequality is violated the diagonal bilinear form becomes degenerate indicating a singularity which a trajectory cannot cross. Therefore, these inequalites put a constraint on allowed trajectories.

It could be argued that a singularity occurs when $B \to -\infty$ as $R^2 - \alpha R + a^2 \to 0$. However, $R^2 - \alpha R + a^2 = R(R-\alpha) + a^2 \geq a^2$ since $R \geq \alpha$. If $a = 0$ the gravitating body is no longer rotating and the Kerr metric no longer applies.

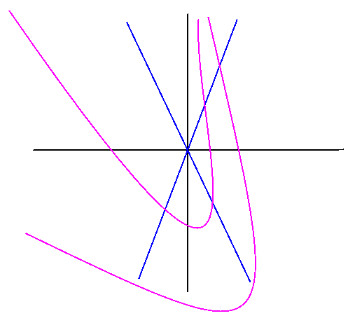
Incidentally, in the equatorial plane relative to the axis of rotation, $cos\theta = 0$ so $\tilde{R} = \frac{R^2+a^2cos^2\theta}{R} = R$. Therefore an object in the equatorial plane does not encounter any singularity until it hits the point mass at the center at $R = \alpha$ and $r = 0$ where its trajectory stops.

The following is a schematic of the different types of trajectories.



The crashed and trapped trajectories require no further discussion except to say that a trapped trajectory which escapes the region of entrapment will crash into the center, not having enough kinetic energy to escape. The trajectories which escape along the axis of rotation have been caught in the gravitation field but cross into the region of entrapment at such an angle and with such a value of kinetic energy that when they encounter the repelling gravitational field within the entrapment region they are expelled, perhaps with escape velocity.

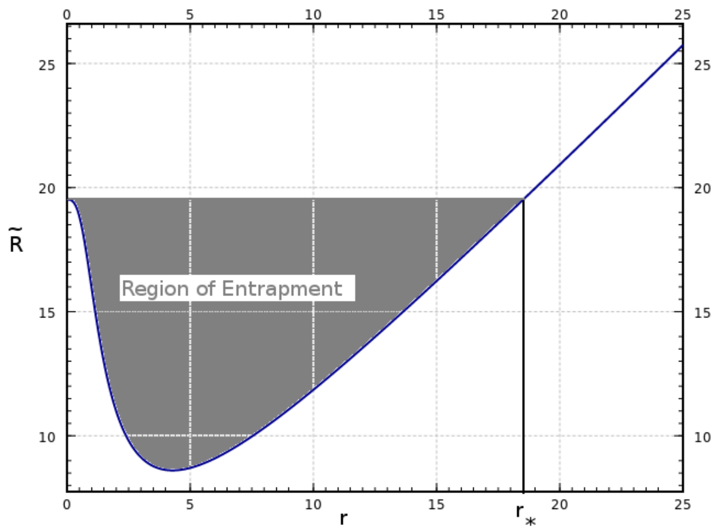
In the following figure we see two such possible trajectories. One passes through the lower entrapment region having sufficient kinetic energy to not be entrapped and then passes into the upper entrapment region and is expelled by the negative field. The other bypasses the lower entrapment region but enters the upper to be expelled.



Above we stated that material caught in the entrapment regions emits electromagnetic radiation. This phenomenon is well observed even with smaller home-based telescopes viewing in the visible wavelengths. In *The Schwarzschild Metric* , while discussing the spherically symmetric case we stated that black holes are black, not because of some event horizon but because light from the collapsed star still being emitted due to its still being very hot is strongly red-shifted and the flux is reduced. There we stated the flux reduction was due to the very intense negative curvature close to the near-singularity and the red-shift was due to time dilation.

For the 'lamps' or 'candlesticks' along the axis of rotation of a Kerr black hole both these effects are reduced. Let r_* be that value such that

$$\frac{\alpha^2 + a^2 cos^2\theta}{\alpha} = \frac{(r_*^3 + \alpha^3)^{2/3} + a^2 cos^2\theta}{(r_*^3 + \alpha^3)^{1/3}}$$



At r_* the negative curvature is greatly reduced since $\tilde{R}/r_* \approx 1$. So, the divergence of flux lines is greatly reduced. At $\tilde{R}(r_*)$ the time dilation is also greatly reduced compared to at the minimum $(R, \tilde{R}) = (|a|cos\theta, 2|a|cos\theta)$. The combined effect would be less flux reduction and less red-shifting of photons being emitted from the hot central mass compared to the region where $cos\theta < \frac{\alpha}{|a|}$. A region of entrapment only exists if $|a| > \alpha$ so

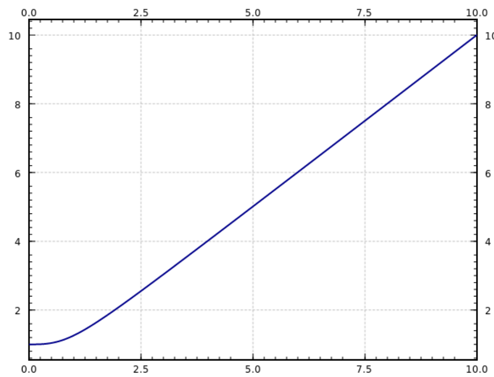
$$\tilde{R}(r_*) = \frac{(r_*^3 + \alpha^3)^{2/3} + a^2cos^2\theta}{(r_*^3 + \alpha^3)^{1/3}} = \frac{\alpha^2 + a^2cos^2\theta}{\alpha} > \frac{\alpha^2 + \alpha^2cos^2\theta}{\alpha}$$

So, at $\theta = 0$, $\tilde{R}(r_*) > 2\alpha$. Then $(1 - \frac{\alpha}{\tilde{R}(r_*)}) > \frac{1}{2}$. So, along the axis of rotation, photons can escape with red-shift $z < 0.414$ measured 'at infinity'.

Inside the region of entrapment, photons increase in energy as they go from $R = \alpha$ to $R = |a|cos\theta$ exciting particles swirling around in the entrapment region causing them to be either expelled within the cone around the axis of rotation or re-emit electromagnetic radiation of various wavelengths or perhaps both.

Gamma Ray bursts:

Suppose an object of mass m is attracted from 'infinity' into the accretion disk and ultimately crashes into the central mass. The energy released will be $E = \frac{GMm}{\alpha}$. Suppose a portion $E' \leq E$ is released immediately as photons emanating in all angular directions. For photons traveling in the angular direction θ such that $cos\theta \leq \frac{\alpha}{a}$, most energy would be rapidly dissipated as red-shift as in the non-rotating case.



However, along the axis of rotation, as has already been shown, the red-shift $z < .414$ measured at 'infinity'. This would allow highly energetic photons to appear as gamma rays across very large expanses of space.

<u>Behavior Along the Axis of Rotation</u>:

On the axis of rotation, $\theta = 0$ so the original metric $K = \begin{pmatrix} (1 - \frac{\alpha}{\tilde{R}}) & 0 & 0 & 0 \\ 0 & -\frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & -\Sigma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The coordinates t, R, θ are othogonal. Then $\frac{d}{ds}[g_{ii}\frac{dx^i}{ds}] = \frac{1}{2}\Sigma_{j=0}^{2}\partial_i g_{jj}\frac{dx^j}{ds}\frac{dx^j}{ds}$ for $0 \leq i \leq 2$ give the geodesic equations of motion with respect to the arclength parameter s along the axis of rotation.

t is the time 'at infinity', $R = (r^3 + \alpha^3)^{1/3}$ is the area radius, $\alpha = 2GM$, and $\tilde{R} = \frac{R^2+a^2}{R}$.

$g_{00} = 1 - \frac{\alpha}{\tilde{R}}, g_{11} = -\frac{\Sigma}{\Delta}$ and $g_{22} = -\Sigma$.

So, $\partial_0 g_{jj} = 0$ for all $j = 0, 1, 2$ so $g_{00}\frac{dx^0}{ds} = constant = E$ where E is the total energy and can be identified as the Hamiltonian of the system.

For convenience in what follows let $\kappa_{\tilde{R}} = 1 - \frac{\alpha}{\tilde{R}}$

For a photon following a geodesic path

$\frac{d\gamma}{ds} = \frac{dt}{ds}\partial_t + \frac{dR}{ds}\partial_R + \frac{d\theta}{ds}\partial_\theta + \frac{d\phi}{ds}\partial_\phi$

$ds^2 = (1 - \frac{\alpha}{\tilde{R}})dt^2 - \frac{\Sigma}{\Delta}dR^2$

The inner product $0 = \langle \frac{d\gamma}{ds}, \frac{d\gamma}{ds} \rangle$

$= (\frac{dt}{ds})^2\langle \partial_t, \partial_t \rangle + (\frac{dR}{ds})^2\langle \partial_R, \partial_R \rangle + (\frac{d\theta}{ds})^2\langle \partial_\theta, \partial_\theta \rangle + (\frac{d\phi}{ds})^2\langle \partial_\phi, \partial_\phi \rangle$

$= (E/\kappa_{\tilde{R}})^2\kappa_{\tilde{R}} - (\frac{dR}{ds})^2\frac{\Sigma}{\Delta}$

$= E^2\kappa_{\tilde{R}}^{-1} - (\frac{dR}{ds})^2\frac{\Sigma}{\Delta}$

Then

$E^2\kappa_{\tilde{R}}^{-1} = (\frac{dR}{ds})^2\frac{\Sigma}{\Delta}$

$E^2 = (\frac{dR}{ds})^2\kappa_{\tilde{R}}\frac{\Sigma}{\Delta} = (\frac{dR}{ds})^2(1 - \frac{\alpha}{\tilde{R}})\frac{R^2+a^2}{R^2-\alpha R+a^2}$ is the Energy Equation for a photon traveling along the axis of rotation. Notice that in the limiting case where $a = 0$ this reduces to the energy equation for the spherically symmetric case along radial lines where $L = 0$, $E^2 = (\frac{dR}{ds})^2$.

To compute the energy equation for a material particle of unit mass we replace s with proper time τ and then $1 = \langle \frac{d\gamma}{d\tau}, \frac{d\gamma}{d\tau} \rangle$ giving the energy equation

$E^2 = (\frac{dR}{d\tau})^2\kappa_{\tilde{R}}\frac{\Sigma}{\Delta} + \kappa_{\tilde{R}} = (1 - \frac{\alpha}{\tilde{R}})[1 + (\frac{dR}{d\tau})^2\frac{R^2+a^2}{R^2-\alpha R+a^2}]$.

In the limiting case where $a = 0$, this reduces to the spherically symmetric case along radial lines where $L = 0$, $E^2 = (\frac{dR}{d\tau})^2 + \kappa_R$.

A unit mass with just enough energy to escape to infinity has $E = 1$. So, we set

$$E^2 = (1 - \frac{\alpha}{\tilde{R}})[1 + (\frac{dR}{d\tau})^2\frac{R^2 + a^2}{R^2 - \alpha R + a^2}] = 1$$

. Then

$$(\frac{dR}{d\tau})^2\frac{R^2 + a^2}{R^2 - \alpha R + a^2} - \frac{\alpha}{\tilde{R}} - \frac{\alpha}{\tilde{R}}(\frac{dR}{d\tau})^2\frac{R^2 + a^2}{R^2 - \alpha R + a^2} = 0$$

and

$$(\frac{dR}{d\tau})^2 \frac{R^2 + a^2}{R^2 - \alpha R + a^2} - \frac{\alpha}{\tilde{R}} (\frac{dR}{d\tau})^2 \frac{R^2 + a^2}{R^2 - \alpha R + a^2} = \frac{\alpha}{\tilde{R}}$$

and

$$(1 - \frac{\alpha}{\tilde{R}})[(\frac{dR}{d\tau})^2 \frac{R^2 + a^2}{R^2 - \alpha R + a^2}] = \frac{\alpha}{\tilde{R}}$$

Consequently

$$(\frac{dR}{d\tau})^2 = \frac{\alpha}{\tilde{R} - \alpha} \frac{R^2 - \alpha R + a^2}{R^2 + a^2}$$

Therefore the unit mass can escape at sub-luminal velocities if

$$\frac{\alpha}{\tilde{R} - \alpha} < \frac{R^2 + a^2}{R^2 - \alpha R + a^2}$$

This allows rotating 'black holes' to 'burp' along or near the axis of rotation.

Consider for a moment the spherically symmetric case. A unit one mass has energy $E^2 = (\frac{dR}{d\tau})^2 + \kappa_R$. If it comes to rest ($\frac{dR}{d\tau} = 0$) at infinity then its energy must equal 1. Then at $R = \alpha$, $\kappa_R = 0$ and $\frac{dR}{d\tau} = 1$ (luminal velocity).

Now consider an escape velocity from $(R, \tilde{R}) = (|a|, 2|a|)$ at the lowest point of the energy well along the axis of rotation. As in the previous case $E = 1$. Then $1 = (1 - \frac{\alpha}{2|a|})[1 + (\frac{dR}{d\tau})^2 \frac{2a^2}{2a^2 - \alpha|a|}] = 1 - \frac{\alpha}{2|a|} + (\frac{dR}{d\tau})^2$. Consequently, $(\frac{dR}{d\tau})^2 = \frac{\alpha}{2|a|}$ indicating a subluminal velocity under the assumption that $\frac{\alpha}{2} < |a|$.

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Schwarzschild, Karl, On the gravitational field of a mass point according to Einstein's theory, english translation by Antoci, S.; Loinger, A. (1999)

Kerr, Roy, Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics, Physical Review Letters. 11 (5): 237–238 (1963)

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