Quaternion Space-time (Part 2):

Pauli Matrices and Dirac Matrices

The Pauli matrices are defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Their multiplication table is:

$\times$	I	σ_1	σ_2	σ_3
I	I	σ_1	σ_2	σ_3
σ_1	σ_1	I	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	I	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	I

There is an isomorphism from the span of the quaternions $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to the span of $\{I, i\sigma_1, i\sigma_2, i\sigma_3,\}$ given by $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (I, -i\sigma_1, -i\sigma_2, -i\sigma_3)$ or by $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (I, i\sigma_3, i\sigma_2, i\sigma_1)$.

The Lie group SU(2) given by $\{a\mathbf{1} + b\mathbf{i} - c\mathbf{j} + d\mathbf{k} : a^2 + b^2 + c^2 + d^2 = 1\}$ is diffeomorphic to $S^3 = \{a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a^2 + b^2 + c^2 + d^2 = 1\}$ and has as generators the set $\{\mathbf{i}, -\mathbf{j}, \mathbf{k}\} \simeq \{-i\sigma_1, i\sigma_2, -i\sigma_3\}$. SU(2) is used in the description of electroweak interactions and beta decay.

The Dirac matrices (in contravariant form) are:

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

Their multiplication table is:

$\times$	γ^0	γ^1	γ^2	γ^3
γ^0	I	$\gamma^0\gamma^1$	$\gamma^0\gamma^2$	$\gamma^0\gamma^3$
γ^1	$-\gamma^0\gamma^1$	$-I$	$-i\sigma_3 I$	$i\sigma_2 I$
γ^2	$-\gamma^0\gamma^2$	$i\sigma_3 I$	$-I$	$-i\sigma_1 I$
γ^3	$-\gamma^0\gamma^3$	$-i\sigma_2 I$	$i\sigma_1 I$	$-I$

and in terms of quaternions

$\times$	γ^0	γ^1	γ^2	γ^3
γ^0	I	$\gamma^0\gamma^1$	$\gamma^0\gamma^2$	$\gamma^0\gamma^3$
γ^1	$-\gamma^0\gamma^1$	$-I$	$\mathbf{k}I$	$-\mathbf{j}I$
γ^2	$-\gamma^0\gamma^2$	$-\mathbf{k}I$	$-I$	$\mathbf{i}I$
γ^3	$-\gamma^0\gamma^3$	$\mathbf{j}I$	$-\mathbf{i}I$	$-I$

The D'Alembert operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ in the relativistic wave equation

$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\psi = 0$

can be expressed as $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = (\frac{1}{c}\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z})^2$.

We can construct a representation of the quaternions in $M_{4\times4}(\mathbf{C})$:

$(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) = (I, \gamma^2\gamma^3, \gamma^3\gamma^1, \gamma^1\gamma^2)$

It is not difficult to check that this construction produces the required properties of quaternions.

Consider the following matrices in $M_{4\times4}\{0, \pm1\}^*$:

$$E_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$E_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Their multiplication table is

$\times$	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_0	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_1	E_1	$-E_0$	E_3	$-E_2$	E_5	$-E_4$	E_7	$-E_6$
E_2	E_2	E_3	$-E_0$	$-E_1$	E_6	E_7	$-E_4$	$-E_5$
E_3	E_3	$-E_2$	$-E_1$	E_0	E_7	$-E_6$	$-E_5$	E_4
E_4	E_4	E_5	$-E_6$	$-E_7$	$-E_0$	$-E_1$	E_2	E_3
E_5	E_5	$-E_4$	$-E_7$	E_6	$-E_1$	E_0	E_3	$-E_2$
E_6	E_6	E_7	E_4	E_5	$-E_2$	$-E_3$	$-E_0$	$-E_1$
E_7	E_7	$-E_6$	E_5	$-E_4$	$-E_3$	E_2	$-E_1$	E_0

We observe that

$\times$	E_0	E_2	E_4	E_6
E_0	E_0	E_2	E_4	E_6
E_2	E_2	$-E_0$	E_6	$-E_4$
E_4	E_4	$-E_6$	$-E_0$	E_2
E_6	E_6	E_4	$-E_2$	$-E_0$

has the same multiplicative structure as the quaternions:

$\times$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

giving the isomorphism $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (E_0, E_2, E_4, E_6)$.

Using the correspondence $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (I, -i\sigma_1, -i\sigma_2, -i\sigma_3)$,

$\times$	E_0	E_3	E_5	E_7
E_0	E_0	E_3	E_5	E_7
E_3	E_3	E_0	$-E_6$	E_4
E_5	E_5	E_6	E_0	$-E_2$
E_7	E_7	$-E_4$	E_2	E_0

has the same multiplicative structure as the Pauli matrices:

$\times$	I	σ_1	σ_2	σ_3
I	I	σ_1	σ_2	σ_3
σ_1	σ_1	I	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	I	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	I

giving the isomorphism $(I, \sigma_1, \sigma_2, \sigma_3) \leftrightarrow (E_0, E_3, E_5, E_7)$.

The group $\mathbf{E} = \{\pm E_0, \pm E_1, \pm E_2, \pm E_3, \pm E_4, \pm E_5, \pm E_6, \pm E_7\}$ contains the Pauli matrices $\{E_3, E_5, E_7\}$ as a subset and the quaternion group $\{\pm E_0, \pm E_2, \pm E_4, \pm E_6\}$ as a subgroup. Any group containing the Pauli matrices must also contain the quaternion group as a subgroup. E is the smallest such group with that property.

Since E_1 commutes with the other E's we can write

$\times$	E_1E_3	E_1E_5	E_1E_7
E_1E_3	$-E_0$	E_6	$-E_4$
E_1E_5	$-E_6$	$-E_0$	E_2
E_1E_7	E_4	$-E_2$	$-E_0$

Then using the following table:

$\times$	γ^0	γ^1	γ^2	γ^3
γ^0	I	$\gamma^0\gamma^1$	$\gamma^0\gamma^2$	$\gamma^0\gamma^3$
γ^1	$-\gamma^0\gamma^1$	$-I$	$\mathbf{k}I$	$-\mathbf{j}I$
γ^2	$-\gamma^0\gamma^2$	$-\mathbf{k}I$	$-I$	$\mathbf{i}I$
γ^3	$-\gamma^0\gamma^3$	$\mathbf{j}I$	$-\mathbf{i}I$	$-I$

we observe there is an isomorphism $(E_0, E_1E_3, E_1E_5, E_1E_7) \leftrightarrow (I, \gamma^1, \gamma^2, \gamma^3)$.

The group $\mathbf{F} = \{\pm\gamma^0, \pm\gamma^1, \pm\gamma^2, \pm\gamma^3, \pm\gamma^0\gamma^1, \pm\gamma^0\gamma^2, \pm\gamma^0\gamma^3, \pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ contains the Dirac matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ as a subset and the quaternion group $\{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ as a subgroup. Any group containing the Dirac matrices must also contain the quaternion group as a subgroup. F is the smallest such group with that property.

Note that $\mathbf{E}$ can be written as $\mathbf{E} = \{\pm E_0, \pm E_1, \pm E_2, \pm E_3, \pm E_4, \pm E_5, \pm E_6, \pm E_7\}$ $= \{\pm\mathbf{1}, \pm i\mathbf{1}, \pm\mathbf{i}, \pm i\mathbf{i}, \pm\mathbf{j}, \pm i\mathbf{j}, \pm\mathbf{k}, \pm i\mathbf{k}\}$.

We can find the solutions of the Dirac equation in terms of the Es defined above.

Since $E_1^2 = -E_0$, $\hat{E} = E_1\partial_0$ is the energy operator and $\hat{p}_i = -E_1\partial_i$ is the i-th momentum operator, the Dirac Hamiltonian can be written as

$$H = -\Sigma_{n=3,5,7}\begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix}\begin{pmatrix} \hat{p}_{(n-1)/2} \\ \hat{p}_{(n-1)/2} \end{pmatrix} + m\begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix}$$

Recall that $-E_3E_1 = E_2$, $-E_5E_1 = E_4$, and $-E_7E_1 = E_6$. Assume $H\psi_L = \hat{E}\psi_L$ and $H\psi_R = -\hat{E}\psi_R$. Then setting $H\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix}\begin{pmatrix} \partial_0\psi_L \\ \partial_0\psi_R \end{pmatrix}$, we can write the Dirac equation as

$$\begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix}\begin{pmatrix} \partial_0\psi_L \\ \partial_0\psi_R \end{pmatrix}$$
$$+\Sigma_{n=2,4,6}\begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix}\begin{pmatrix} \partial_{n/2}\psi_L \\ \partial_{n/2}\psi_R \end{pmatrix} - m\begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \mathbf{0}$$

which gives us the two equations:

$$\begin{aligned} E_1\partial_0\psi_L - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_R &= mE_0\psi_L \\ E_1\partial_0\psi_R - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_L &= -mE_0\psi_R \end{aligned}$$

Since $E_1\partial_0$ is the energy operator the above equations are equivalent to

$$E\psi_L - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_R = m\psi_L \quad (1)$$

$$E\psi_R - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_L = -m\psi_R \quad (2)$$

Then

$$\psi_L = \frac{\Sigma_{n=2,4,6}E_n\partial_{n/2}}{E-m}\psi_R \quad (3)$$

$$\psi_R = \frac{\Sigma_{n=2,4,6}E_n\partial_{n/2}}{E+m}\psi_L \quad (4)$$

Writing $\eta = \Sigma_{n=2,4,6}E_n\partial_{n/2}$ and substituting (4) into (1):

$(E-m)\psi_L - \eta\frac{\eta}{E+m}\psi_L = 0$

$\eta^2 = (E_2\partial_x + E_4\partial_y + E_6\partial_z)^2 = -\partial_x^2 - \partial_y^2 - \partial_z^2$

and since $-E_1\partial_i = \hat{p}_i$ and $E_1^2 = -E_0$ we have $\eta^2 = \hat{p}^2$

$(E-m)\psi_L - \eta\frac{\eta}{E+m}\psi_L = 0$ implies

$(E-m)\psi_L - \frac{\hat{p}^2}{E+m}\psi_L = 0.$

Then $[(E+m)(E-m) - p^2]\psi_L = 0.$

So, $(E^2 - p^2 - m^2)\psi_L = 0$ and similarly $(E^2 - p^2 - m^2)\psi_R = 0$. Since the wave functions are assumed to be non-zero, $E^2 - p^2 = m^2$, which is consistent with the 4-vector magnitude-squared.

Recall that $-E_3E_1 = E_2$, $-E_5E_1 = E_4$, and $-E_7E_1 = E_6$ so we can re-write equations (3) and (4) as:

$$\psi_L = \frac{-\Sigma_{n=3,5,7}E_nE_1\partial_{(n-1)/2}}{E-m}\psi_R \quad (5)$$

$$\psi_R = \frac{-\Sigma_{n=3,5,7}E_nE_1\partial_{(n-1)/2}}{E+m}\psi_L \quad (6)$$

Recall that $-E_1\partial_i = \hat{p}_i$ so we can re-write equations (5) and (6) as:

$$\psi_L = \frac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E-m}\psi_R \quad (7)$$

$$\psi_R = \frac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E+m}\psi_L \quad (8)$$

We can write $\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}$ as:

$$\Pi = \begin{pmatrix} \hat{p}_z & 0 & \hat{p}_x & \hat{p}_y \\ 0 & \hat{p}_z & -\hat{p}_y & \hat{p}_x \\ \hat{p}_x & -\hat{p}_y & -\hat{p}_z & 0 \\ \hat{p}_y & \hat{p}_x & 0 & -\hat{p}_z \end{pmatrix}$$

Then we have the following products:

$$\Pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{p}_z & 0 \\ 0 & \hat{p}_z \\ \hat{p}_x & -\hat{p}_y \\ \hat{p}_y & \hat{p}_x \end{pmatrix} \text{ and } \Pi \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{p}_x & \hat{p}_y \\ -\hat{p}_y & \hat{p}_x \\ -\hat{p}_z & 0 \\ 0 & -\hat{p}_z \end{pmatrix}$$

For $E \geq m$ two linearly independent solutions are

$$\phi = [\tfrac{1}{E+m} \begin{pmatrix} p_z & 0 \\ 0 & p_z \\ p_x & -p_y \\ p_y & p_x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}] e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

and

$$\phi = [\tfrac{1}{E+m} \begin{pmatrix} p_x & p_y \\ -p_y & p_x \\ -p_z & 0 \\ 0 & -p_z \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}] e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

For $E \leq -m$ two linearly independent solutions are

$$\phi = [\tfrac{1}{E-m} \begin{pmatrix} p_z & 0 \\ 0 & p_z \\ p_x & -p_y \\ p_y & p_x \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}] e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

and

$$\phi = [\tfrac{1}{E-m} \begin{pmatrix} p_x & p_y \\ -p_y & p_x \\ -p_z & 0 \\ 0 & -p_z \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}] e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ correspond to the two spin states of a particle.

Verification of solutions:

Case $E \geq m$ with spin '+':

$$\phi = [\tfrac{1}{E+m}\begin{pmatrix} p_z & 0 \\ 0 & p_z \\ p_x & -p_y \\ p_y & p_x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}]e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$[E_1\partial_t - (\Sigma_{n=2,4,6}E_n\partial_{n/2})]\phi$

$$= E_1\partial_t\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\tfrac{1}{E+m}\begin{pmatrix} \hat{p}_z & 0 \\ 0 & \hat{p}_z \\ \hat{p}_x & -\hat{p}_y \\ \hat{p}_y & \hat{p}_x \end{pmatrix}e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= E_1\partial_t\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\tfrac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E+m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= \begin{pmatrix} E & 0 \\ 0 & E \\ 0 & 0 \\ 0 & 0 \end{pmatrix}e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\tfrac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E+m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$= E\psi_L - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_R.$

Furthermore, $E\psi_L - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_R = m\psi_L$

because $(\Sigma_{n=2,4,6}E_n\partial_{n/2})(\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}) = (\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2})^2$

$= \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$

and $E - \frac{p_x^2+p_y^2+p_z^2}{E+m} = m$ since $(E-m)(E+m) - (p_x^2+p_y^2+p_z^2) = 0$.

Case $E \geq m$ with spin '-':

$$\phi = [\tfrac{1}{E+m}\begin{pmatrix} p_x & p_y \\ -p_y & p_x \\ -p_z & 0 \\ 0 & -p_z \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}]e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$[E_1\partial_t - (\Sigma_{n=2,4,6}E_n\partial_{n/2})]\phi$

$$= E_1\partial_t \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\tfrac{1}{E+m}\begin{pmatrix} \hat{p}_x & \hat{p}_y \\ -\hat{p}_y & \hat{p}_x \\ -\hat{p}_z & 0 \\ 0 & -\hat{p}_z \end{pmatrix} e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= E_1\partial_t \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\tfrac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E+m}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ E & 0 \\ 0 & E \end{pmatrix} e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\tfrac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E+m}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$= E\psi_L - (\Sigma_{n=2,4,6} E_n \partial_{n/2})\psi_R.$

Furthermore, $E\psi_L - (\Sigma_{n=2,4,6} E_n \partial_{n/2})\psi_R = m\psi_L$

because $(\Sigma_{n=2,4,6} E_n \partial_{n/2})(\Sigma_{n=3,5,7} E_n \hat{p}_{(n-1)/2}) = (\Sigma_{n=3,5,7} E_n \hat{p}_{(n-1)/2})^2$

$= \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$

and $E - \frac{p_x^2+p_y^2+p_z^2}{E+m} = m$ since $(E-m)(E+m) - (p_x^2 + p_y^2 + p_z^2) = 0$.

Case $E \leq -m$ with spin '+':

$$\phi = [\tfrac{1}{E-m}\begin{pmatrix} p_z & 0 \\ 0 & p_z \\ p_x & -p_y \\ p_y & p_x \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}]e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$[E_1\partial_t - (\Sigma_{n=2,4,6} E_n \partial_{n/2})]\phi$

$$= -E_1\partial_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6} E_n \partial_{n/2})\tfrac{1}{E-m}\begin{pmatrix} \hat{p}_z & 0 \\ 0 & \hat{p}_z \\ \hat{p}_x & -\hat{p}_y \\ \hat{p}_y & \hat{p}_x \end{pmatrix} e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= -E_1\partial_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6} E_n \partial_{n/2})\frac{\Sigma_{n=3,5,7} E_n \hat{p}_{(n-1)/2}}{E-m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= \begin{pmatrix} E & 0 \\ 0 & E \\ 0 & 0 \\ 0 & 0 \end{pmatrix} e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\frac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E-m}\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= E\psi_R - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_L.$$

Furthermore, $E\psi_R - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_L = -m\psi_R$

because $(\Sigma_{n=2,4,6}E_n\partial_{n/2})(\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}) = (\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2})^2$

$= \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$

and $E - \frac{p_x^2+p_y^2+p_z^2}{E-m} = -m$ since $(E+m)(E-m) - (p_x^2+p_y^2+p_z^2) = 0.$

Case $E \leq -m$ with spin '-':

$$\phi = [\frac{1}{E-m}\begin{pmatrix}p_x&p_y\\-p_y&p_x\\-p_z&0\\0&-p_z\end{pmatrix} - \begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}]e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$[E_1\partial_t - (\Sigma_{n=2,4,6}E_n\partial_{n/2})]\phi$

$$= -E_1\partial_t\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\frac{1}{E-m}\begin{pmatrix}\hat{p}_x&\hat{p}_y\\-\hat{p}_y&\hat{p}_x\\-\hat{p}_z&0\\0&-\hat{p}_z\end{pmatrix}e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= -E1\partial_t\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\frac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E-m}\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ E & 0 \\ 0 & E \end{pmatrix} e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$-(\Sigma_{n=2,4,6}E_n\partial_{n/2})\frac{\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}}{E-m}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} e^{E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

$$= E\psi_R - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_L.$$

Furthermore, $E\psi_R - (\Sigma_{n=2,4,6}E_n\partial_{n/2})\psi_L = -m\psi_R$

because $(\Sigma_{n=2,4,6}E_n\partial_{n/2})(\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2}) = (\Sigma_{n=3,5,7}E_n\hat{p}_{(n-1)/2})^2$

$= \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$

and $E - \frac{p_x^2+p_y^2+p_z^2}{E-m} = -m$ since $(E+m)(E-m) - (p_x^2+p_y^2+p_z^2) = 0$.**

Lorentz Invariance of Dirac equation:

$$\begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix}\begin{pmatrix} \partial_0\psi_L \\ \partial_0\psi_R \end{pmatrix}$$
$$+\Sigma_{n=2,4,6}\begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix}\begin{pmatrix} \partial_{n/2}\psi_L \\ \partial_{n/2}\psi_R \end{pmatrix} = m\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Setting the Lorentz transform $\Lambda = \Lambda^T = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) \\ sinh(\alpha) & cosh(\alpha) \end{pmatrix}$

we have by direct calculation

$$\Lambda\begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix}\Lambda = \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix}$$

and

$$\Lambda\begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix}\Lambda = \begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix}$$

Then applying the Lorentz transform

$$\Lambda^T \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix} [\Lambda \begin{pmatrix} \partial'_0\psi'_L \\ \partial'_0\psi'_R \end{pmatrix}]$$
$$+\Sigma_{n=2,4,6}\Lambda^T \begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix} [\Lambda \begin{pmatrix} \partial'_{n/2}\psi'_L \\ \partial'_{n/2}\psi'_R \end{pmatrix}] = m \begin{pmatrix} \psi'_L \\ \psi'_R \end{pmatrix}$$

which is equivalent to

$$[\Lambda^T \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix} \Lambda] \begin{pmatrix} \partial'_0\psi'_L \\ \partial'_0\psi'_R \end{pmatrix}$$
$$+\Sigma_{n=2,4,6}[\Lambda^T \begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix} \Lambda] \begin{pmatrix} \partial'_{n/2}\psi'_L \\ \partial'_{n/2}\psi'_R \end{pmatrix} = m \begin{pmatrix} \psi'_L \\ \psi'_R \end{pmatrix}$$

and

$$\begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix} \begin{pmatrix} \partial'_0\psi'_L \\ \partial'_0\psi'_R \end{pmatrix}$$
$$+\Sigma_{n=2,4,6} \begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix} \begin{pmatrix} \partial'_{n/2}\psi'_L \\ \partial'_{n/2}\psi'_R \end{pmatrix} = m \begin{pmatrix} \psi'_L \\ \psi'_R \end{pmatrix}$$

The Dirac Matrices in $M_{8\times 8}\{0, \pm 1\}$:

The Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

or, in terms of 4×4 matrices

$$\sigma_1 = E_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \sigma_2 = E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\sigma_3 = E_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

The Dirac matrices are:

$$\gamma^0 = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} \mathbf{0} & \sigma_1 \\ -\sigma_1 & \mathbf{0} \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} \mathbf{0} & \sigma_2 \\ -\sigma_2 & \mathbf{0} \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} \mathbf{0} & \sigma_3 \\ -\sigma_3 & \mathbf{0} \end{pmatrix}$$

or in terms of 8×8 matrices

$$\gamma^0 = \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & -E_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} \mathbf{0} & E_3 \\ -E_3 & \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} \mathbf{0} & E_5 \\ -E_5 & \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} \mathbf{0} & E_7 \\ -E_7 & \mathbf{0} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Their multiplication table is:

$\times$	γ^0	γ^1	γ^2	γ^3
γ^0	I	$\gamma^0\gamma^1$	$\gamma^0\gamma^2$	$\gamma^0\gamma^3$
γ^1	$-\gamma^0\gamma^1$	$-I$	$\gamma^1\gamma^2$	$\gamma^1\gamma^3$
γ^2	$-\gamma^0\gamma^2$	$-\gamma^1\gamma^2$	$-I$	$\gamma^2\gamma^3$
γ^3	$-\gamma^0\gamma^3$	$-\gamma^1\gamma^3$	$-\gamma^2\gamma^3$	$-I$

Referring back to the table:

$\times$	γ^0	γ^1	γ^2	γ^3
γ^0	I	$\gamma^0\gamma^1$	$\gamma^0\gamma^2$	$\gamma^0\gamma^3$
γ^1	$-\gamma^0\gamma^1$	$-I$	$\mathbf{k}I$	$-\mathbf{j}I$
γ^2	$-\gamma^0\gamma^2$	$-\mathbf{k}I$	$-I$	$\mathbf{i}I$
γ^3	$-\gamma^0\gamma^3$	$\mathbf{j}I$	$-\mathbf{i}I$	$-I$

We can use the current table to construct a representation of the quaternions in $M_{8\times 8}\{0,\pm 1\}$:

$$(\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}) = (I, \gamma^2\gamma^3, \gamma^3\gamma^1, \gamma^1\gamma^2)$$

It is not difficult to check that this construction produces the required properties of quaternions.

Above we wrote the Dirac equation as::

$$\begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & -E_1 \end{pmatrix} \begin{pmatrix} \partial_0\psi_L \\ \partial_0\psi_R \end{pmatrix}$$
$$+\Sigma_{n=2,4,6} \begin{pmatrix} \mathbf{0} & -E_n \\ E_n & \mathbf{0} \end{pmatrix} \begin{pmatrix} \partial_{n/2}\psi_L \\ \partial_{n/2}\psi_R \end{pmatrix} - m \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \mathbf{0}$$

Multiplying through by E_1 and changing sign throughout gives

$$\begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & -E_0 \end{pmatrix} \begin{pmatrix} \partial_0\psi_L \\ \partial_0\psi_R \end{pmatrix}$$
$$+\Sigma_{n=3,5,7} \begin{pmatrix} \mathbf{0} & E_n \\ -E_n & \mathbf{0} \end{pmatrix} \begin{pmatrix} \partial_{(n-1)/2}\psi_L \\ \partial_{(n-1)/2}\psi_R \end{pmatrix}$$
$$+ m \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & E_1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \mathbf{0}$$

which gives the more familiar form of the equation

$$\gamma^\mu \begin{pmatrix} \partial_\mu \psi_L \\ \partial_\mu \psi_R \end{pmatrix} + m \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & E_1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \mathbf{0}$$

Writing this as $(\gamma^0\partial_t+\gamma^1\partial_x+\gamma^2\partial_y+\gamma^3\partial_z) \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = -m \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & E_1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ and applying the operator on both sides gives

$$\begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} (\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2) \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = -m^2 \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

since the gammas anti-commute. A further discussion of the Klein-Gordon equation appears below.

The above gives two equations

$E_0(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2)\psi_L = -m^2 E_0 \psi_L$ and

$E_0(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2)\psi_R = -m^2 E_0 \psi_R$

The equation $(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2)\psi = -m^2\psi$ applies to spin-less particles. However, Dirac particles satisfy the above two equations component-wise.

Proceeding in reverse, so to speak, we begin with

$$\begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} (\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2) \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = -m^2 \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

and factor the operator

$$(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z)^2 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$
$$= -m \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & E_1 \end{pmatrix} (\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z) \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Canceling the operator on each side restores the original Dirac equation

$$(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z) \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = -m \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & E_1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

restoring the spin characteristics of the solutions $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$.

Spinors:

Let ψ_i, $i = 1, 2, 3, 4$ be 2×2 matrix blocks.

For $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ an 8×2 *spinor* we can define its conjugate $\psi^* = \psi^T$ and $\overline{\psi} = \psi^* \gamma^0$.

Then for

$$\psi \in \{ \begin{pmatrix} E & 0 \\ 0 & E \\ 0 & 0 \\ 0 & 0 \\ p_z & 0 \\ 0 & p_z \\ p_x & -p_y \\ p_y & p_x \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ E & 0 \\ 0 & E \\ p_x & p_y \\ -p_y & p_x \\ -p_z & 0 \\ 0 & -p_z \end{pmatrix}, \begin{pmatrix} p_z & 0 \\ 0 & p_z \\ p_x & -p_y \\ p_y & p_x \\ E & 0 \\ 0 & E \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} p_x & p_y \\ -p_y & p_x \\ -p_z & 0 \\ 0 & -p_z \\ 0 & 0 \\ 0 & 0 \\ E & 0 \\ 0 & E \end{pmatrix} \},$$

$\overline{\psi}\psi = \psi^* \gamma^0 \psi = \pm \begin{pmatrix} E^2 - p_x^2 - p_y^2 - p_z^2 & 0 \\ 0 & E^2 - p_x^2 - p_y^2 - p_z^2 \end{pmatrix}$ is Lorentz invariant where $\pm$ corresponds to particle/antiparticle.

Rotation of Spinors:

Let $\begin{pmatrix} a & -b \\ b & a \\ c & -d \\ d & c \end{pmatrix} \sim \begin{pmatrix} a + ib \\ c + id \end{pmatrix}$ be a 4×2 spinor.

furthermore, let $\mathbf{n} = \frac{1}{\sqrt{n_x^2+n_y^2+n_z^2}}(n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k})$ define the unit vector axis of the spinor.

Then $\mathbf{n} = \frac{1}{\sqrt{n_x^2+n_y^2+n_z^2}}(n_x E_2 + n_y E_4 + n_z E_6)$ and the rotation operator is

$R(\theta) = cos(\theta/2)E_0 + sin(\theta/2)(n_x E_2 + n_y E_4 + n_z E_6)$.

Then written out explicitly,

$$R(\theta)\begin{pmatrix} a & -b \\ b & a \\ c & -d \\ d & c \end{pmatrix}$$

$$= \begin{pmatrix} cos(\theta/2) & n_z sin(\theta/2) & -n_y sin(\theta/2) & n_x sin(\theta/2) \\ -n_z sin(\theta/2) & cos(\theta/2) & -n_x sin(\theta/2) & -n_y sin(\theta/2) \\ n_y sin(\theta/2) & n_x sin(\theta/2) & cos(\theta/2) & -n_z sin(\theta/2) \\ -n_x sin(\theta/2) & n_y sin(\theta/2) & n_z sin(\theta/2) & cos(\theta/2) \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \\ c & -d \\ d & c \end{pmatrix}.$$

The Dirac Equation of a Charged Particle in an Electro-magnetic Field:

We can write the equation

$$\gamma^\mu \begin{pmatrix} \partial_\mu \psi_L \\ \partial_\mu \psi_R \end{pmatrix} + m \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & E_1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \mathbf{0}$$

as

$$\gamma^\mu \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & E_1 \end{pmatrix} \begin{pmatrix} \partial_\mu \psi_L \\ \partial_\mu \psi_R \end{pmatrix} - m \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \mathbf{0}$$

We transform the momentum operator $\hat{p}_i \to \hat{p}_i + qE_0 A_i$ where $\mathbf{A}$ is an electromagnetic vector potential. That is, $E_1 \partial_i \to E_1 \partial_i - qE_0 A_i$.

Then we can write $\Sigma_{n=3,5,7} E_n (\hat{p}_{(n-1)/2} + qE_0 A_{(n-1)/2})$ as:

$$\Pi = \begin{pmatrix} \hat{p}_z + qE_0 A_z & 0 & \hat{p}_x + qE_0 A_x & \hat{p}_y + qE_0 A_y \\ 0 & \hat{p}_z + qE_0 A_z & -\hat{p}_y - qE_0 A_y & \hat{p}_x + qE_0 A_x \\ \hat{p}_x + qE_0 A_x & -\hat{p}_y - qE_0 A_y & -\hat{p}_z - qE_0 A_z & 0 \\ \hat{p}_y + qE_0 A_y & \hat{p}_x + qE_0 A_x & 0 & -\hat{p}_z - qE_0 A_z \end{pmatrix}$$

Then we have the following products:

$$\Pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{p}_z + qE_0 A_z & 0 \\ 0 & \hat{p}_z + qE_0 A_z \\ \hat{p}_x + qE_0 A_x & -\hat{p}_y - qE_0 A_y \\ \hat{p}_y - qE_0 A_y & \hat{p}_x + qE_0 A_x \end{pmatrix}$$

and $\Pi \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{p}_x + qE_0A_x & \hat{p}_y + qE_0A_y \\ -\hat{p}_y - qE_0A_y & \hat{p}_x + qE_0A_x \\ -\hat{p}_z - qE_0A_z & 0 \\ 0 & -\hat{p}_z - qE_0A_z \end{pmatrix}$

For $E \geq m$ two linearly independent solutions are

$$\phi = [\tfrac{1}{E+m} \begin{pmatrix} p_z + qA_z & 0 \\ 0 & p_z + qA_z \\ p_x + qA_x & -p_y - qA_y \\ p_y + qA_y & p_x + qA_x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}] e^{-E_1(Et - \mathbf{p}\cdot\mathbf{r})}$$

and

$$\phi = [\tfrac{1}{E+m} \begin{pmatrix} p_x + qA_x & p_y + qA_y \\ -p_y - qA_y & p_x + qA_x \\ -p_z - qA_z & 0 \\ 0 & -p_z - qA_z \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}] e^{-E_1(Et - \mathbf{p}\cdot\mathbf{r})}$$

For $E \leq -m$ two linearly independent solutions are

$$\phi = [\tfrac{1}{E-m} \begin{pmatrix} p_z + qA_z & 0 \\ 0 & p_z + qA_z \\ p_x + qA_x & -p_y - qA_y \\ p_y + qA_y & p_x + qA_x \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}] e^{E_1(Et - \mathbf{p}\cdot\mathbf{r})}$$

and

$$\phi = [\tfrac{1}{E-m} \begin{pmatrix} p_x + qA_x & p_y + qA_y \\ -p_y - qA_y & p_x + qA_x \\ -p_z - qA_z & 0 \\ 0 & -p_z - qA_z \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}] e^{E_1(Et - \mathbf{p}\cdot\mathbf{r})}$$

As in classic E-M theory, the gauge can be varied.

Let $A_i \to A'_i = A_i + \partial_i \chi(x)$. Then the solution $\psi' = e^{-E_1 q\chi}\psi$ since

$(E_1\partial_i - qE_0A'_i)\psi' = qE_0\partial_i\chi\psi' + e^{-E_1q\chi}E_1\partial_i\psi - qE_0A'_i\psi'$

$= e^{-E_1q\chi}qE_0\partial_i\chi\psi + e^{-E_1q\chi}E_1\partial_i\psi - e^{-E_1q\chi}qE_0(A_i + \partial_i\chi)\psi$

$= e^{-E_1q\chi}[qE_0\partial_i\chi + E_1\partial_i - qE_0A_i - qE_0\partial_i\chi)]\psi$

$= e^{-E_1q\chi}[E_1\partial_i - qE_0A_i]\psi$

So, the equation

$$\gamma^\mu \left[\begin{pmatrix} E_1\partial_\mu & \mathbf{0} \\ \mathbf{0} & E_1\partial_\mu \end{pmatrix} - qA'_\mu \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} \right] \begin{pmatrix} \psi'_L \\ \psi'_R \end{pmatrix} - m \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} \begin{pmatrix} \psi'_L \\ \psi'_R \end{pmatrix} = \mathbf{0}$$

is equivalent to

$$\gamma^\mu \left[\begin{pmatrix} E_1\partial_\mu & \mathbf{0} \\ \mathbf{0} & E_1\partial_\mu \end{pmatrix} - qA_\mu \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} \right] \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} - m \begin{pmatrix} E_0 & \mathbf{0} \\ \mathbf{0} & E_0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \mathbf{0}$$

$e^{-E_1 q\chi}$ traces out a circular path around a torus and is isomorphic to S^1. The group $\{e^{-E_1\theta} : \theta \in \mathbf{R}(mod 2\pi)\} \sim U(1)$.

Isospin:

The theory of isospin uses the same mathematics as that developed for intrinsic spin.

Recall the Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and their multiplication table is:

$\times$	I	σ_1	σ_2	σ_3
I	I	σ_1	σ_2	σ_3
σ_1	σ_1	I	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	I	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	I

Let $2\tau_1 = \sigma_1$, $2\tau_2 = \sigma_2$, and $2\tau_3 = \sigma_3$.

Then the τ multiplication table is

$\times$	I	$2\tau_1$	$2\tau_2$	$2\tau_3$
I	I	$2\tau_1$	$2\tau_2$	$2\tau_3$
$2\tau_1$	$2\tau_1$	I	$2i\tau_3$	$-2i\tau_2$
$2\tau_2$	$2\tau_2$	$-2i\tau_3$	I	$2i\tau_1$
$2\tau_3$	$2i\tau_3$	$2i\tau_2$	$-2i\tau_1$	I

Just as for intrinsic spin we have $S = \frac{1}{2}\sigma$ where $\sigma = \sigma_1 + \sigma_2 + \sigma_3$ and $S^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = \frac{3}{4}I$ we also have $T = \tau$ where $\tau = \tau_1 + \tau_2 + \tau_3$ and $T^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = \frac{3}{4}I$.

Let $p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represent a proton and $n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent a neutron. They are to be regarded as the same particle in different isospin states.

The operator $2\tau_1$ transforms between the two isospin states

that is, $2\tau_1 p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = n$ and $2\tau_1 n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = p$

We can calculate the effects of τ_2, and τ_3 on n and p.

$2\tau_2 n = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -ip$ and $2\tau_2 p = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = in$

$2\tau_3 n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -n$ and $2\tau_3 p = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = p$

We can form raising and lowering operators as

$\tau_+ = \tau_1 + i\tau_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\tau_- = \tau_1 - i\tau_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Clearly, $\tau_+ n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = p$

and $\tau_- p = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = n$

The Klein-Gordon Equation:

The *Klein-Gordon* equation $(\partial_t^2 - \nabla^2)\psi = -m^2\psi$ has plane wave solutions of the form $\psi = e^{-i(Et - \mathbf{p}\cdot\mathbf{r})}$ since $\partial_t\psi = -iE\psi$, $\partial_x\psi = ip_x\psi$, $\partial_y\psi = ip_y\psi$, and $\partial_z\psi = ip_z\psi$. Then $(\partial_t^2 - \nabla^2)\psi = (-E^2 + p_x^2 + p_y^2 + p_z^2)\psi = -m^2\psi$ which is the 4-vector magnitude-squared $E^2 - p_x^2 - p_y^2 - p_z^2 = m^2$.

Expressed somewhat differently,

$$(\partial_t^2 - \nabla^2)\psi = \begin{pmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}\psi$$

$$= \begin{pmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -iE\psi \\ ip_x\psi \\ ip_y\psi \\ ip_z\psi \end{pmatrix}$$

$$= \begin{pmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{pmatrix} \begin{pmatrix} -iE\psi \\ -ip_x\psi \\ -ip_y\psi \\ -ip_z\psi \end{pmatrix}$$

$$= ((-iE)^2 + (-ip_x)(ip_x) + (-ip_y)(ip_y) + (-ip_z)(ip_z))\psi$$

$= (-E^2 + p_x^2 + p_y^2 + p_z^2)\psi = -m^2\psi$. which is consistent with the 4-vector magnitude-squared $E^2 - p_x^2 - p_y^2 - p_z^2 = m^2$. This describes an unbound spinless particle with mass m.

The Klein-Gordon equation is invariant under a Lorentz transformation Λ since it preserves the Minkowski metric. That is,

$$\begin{pmatrix} \partial_{t'} & \partial_{x'} & \partial_{y'} & \partial_{z'} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_{t'} \\ \partial_{x'} \\ \partial_{y'} \\ \partial_{z'} \end{pmatrix} \psi$$

$$= [\Lambda \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}]^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \psi$$

$$= \begin{pmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{pmatrix} \Lambda^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \psi$$

$$= \begin{pmatrix} \partial_t & \partial_x & \partial_y & \partial_z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \psi$$

The D'Alembert operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ in the relativistic wave equation

$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\psi = 0$

can be expressed as $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = (\frac{1}{c}\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z})^2$.

Suppose $(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\psi = k^2\psi = -\frac{m^2c^2}{\hbar^2}\psi$

Then the solution must satisfy

$(\frac{1}{c}\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z})\psi = \pm k\psi$ since

$(\frac{1}{c}\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z})^2\psi$

$= (\frac{1}{c}\gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z})(\pm k\psi) = k^2\psi.$

Then $(\gamma^0\frac{\partial}{\partial t} + c\gamma^1\frac{\partial}{\partial x} + c\gamma^2\frac{\partial}{\partial y} + c\gamma^3\frac{\partial}{\partial z})\psi = \pm i\frac{mc}{\hbar}\psi.$

Setting $c = 1$, we get the Dirac equation(s) $(i\hbar\gamma^\mu\partial_\mu \mp m)\psi = 0$ discussed previously.

The D'Alembert operator can also be expressed as

$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \simeq (\mathbf{1}\frac{\partial}{c\partial t})^2 + (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})^2$ or with $c = 1$

$$\frac{\partial^2}{\partial t^2} - \nabla^2 \simeq (\mathbf{1}\frac{\partial}{\partial t})^2 + (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})^2$$

$$= \begin{pmatrix} \partial/\partial_t & \partial/\partial_x & \partial/\partial_y & \partial/\partial_z \end{pmatrix} \begin{pmatrix} \mathbf{11} & 0 & 0 & 0 \\ 0 & \mathbf{ii} & 0 & 0 \\ 0 & 0 & \mathbf{jj} & 0 \\ 0 & 0 & 0 & \mathbf{kk} \end{pmatrix} \begin{pmatrix} \partial/\partial_t \\ \partial/\partial_x \\ \partial/\partial_y \\ \partial/\partial_z \end{pmatrix}$$

We use the covariant derivative to get the general wave equation operator:

$$\begin{pmatrix} \nabla_t & \nabla_u & \nabla_v & \nabla_w \end{pmatrix} \mathbf{G}^{-1} \begin{pmatrix} \nabla_t \\ \nabla_u \\ \nabla_v \\ \nabla_w \end{pmatrix}$$

where $\mathbf{G}^{-1} = \begin{pmatrix} \langle\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\rangle^{-1}\mathbf{11} & 0 & 0 & 0 \\ 0 & \langle\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\rangle^{-1}\mathbf{e}_u\mathbf{e}_u & 0 & 0 \\ 0 & 0 & \langle\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\rangle^{-1}\mathbf{e}_v\mathbf{e}_v & 0 \\ 0 & 0 & 0 & \langle\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\rangle^{-1}\mathbf{e}_w\mathbf{e}_w \end{pmatrix}$

and $\mathbf{G} = \begin{pmatrix} \langle\frac{\partial}{\partial t},\frac{\partial}{\partial t}\rangle\mathbf{11} & 0 & 0 & 0 \\ 0 & \langle\frac{\partial}{\partial u},\frac{\partial}{\partial u}\rangle\mathbf{e}_u\mathbf{e}_u & 0 & 0 \\ 0 & 0 & \langle\frac{\partial}{\partial v},\frac{\partial}{\partial v}\rangle\mathbf{e}_v\mathbf{e}_v & 0 \\ 0 & 0 & 0 & \langle\frac{\partial}{\partial w},\frac{\partial}{\partial w}\rangle\mathbf{e}_w\mathbf{e}_w \end{pmatrix}$

and where the coordinates t, u, v, w are associated with the quaternion basis $\mathbf{1}, \mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w$.

Recall from *Quaternion Space-Time* that for a 4-vector $\mathbf{V}$,

$||\mathbf{V}||^2 = \mathbf{V}^T\mathbf{G}\mathbf{V}$.

The wave equation operator is most easily calculated using the 4D form of the Laplacian:

$\nabla^2 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$.

For R-W spherical coordinates,

where $\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -R^2 & 0 & 0 \\ 0 & 0 & -R^2sin^2\chi & 0 \\ 0 & 0 & 0 & -R^2sin^2\chi sin^2\theta \end{pmatrix}\mathbf{1}$

Then $\frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}g^{tt}\partial_t) = \partial_t^2 + \frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g})\partial_t = 6R^{-1}\dot{R}\partial_t + \partial_t^2$

and for S^3, $\nabla^2_{S^3} = \frac{1}{\sqrt{\eta}}\partial_i(\sqrt{\eta}\eta^{ij}\partial_j) = \frac{1}{\sqrt{\eta}}[\partial_\chi(\sqrt{\eta}\eta^{\chi\chi}\partial_\chi) + \partial_\theta(\sqrt{\eta}\eta^{\theta\theta}\partial_\theta) + \partial_\phi(\sqrt{\eta}g^{\phi\phi}\partial_\phi)]$

where $\eta_{ij} = \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2sin^2\chi & 0 \\ 0 & 0 & R^2sin^2\chi sin^2\theta \end{pmatrix}$

and $\eta = Det(\eta_{ij}) = R^6sin^4\chi sin^2\theta$

and $\sqrt{\eta} = \sqrt{Det(\eta_{ij})} = R^3sin^2\chi sin\theta$

Then

$\nabla^2_{S^3} = \frac{1}{\sqrt{g}}[2R^3sin\chi cos\chi sin\theta R^{-2}\partial_\chi + R^3sin^2\chi sin\theta R^{-2}\partial^2_\chi$

$+R^3 sin^2\chi cos\theta(R^2 sin^2\chi)^{-1}\partial_\theta + R^3 sin^2\chi sin\theta(R^2 sin^2\chi)^{-1}\partial_\theta^2$

$+R^3 sin^2\chi sin\theta(R^2 sin^2\chi sin^2\theta)^{-1}\partial_\phi^2]$

$\nabla_{S^3}^2 = \frac{1}{\sqrt{g}}[2Rsin\chi cos\chi sin\theta\partial_\chi + Rsin^2\chi sin\theta\partial_\chi^2$

$+Rcos\theta\partial_\theta + Rsin\theta\partial_\theta^2 + Rcsc\theta\partial_\phi^2]$

$\nabla_{S^3}^2 = \frac{1}{R^2 sin^2\chi sin\theta}[2sin\chi cos\chi sin\theta\partial_\chi + sin^2\chi sin\theta\partial_\chi^2 + sin\chi cos\theta\partial_\theta + sin\theta\partial_\theta^2$

$+csc\theta\partial_\phi^2]$

$= \frac{1}{R^2}[2cot\chi\partial_\chi + \partial_\chi^2 + csc\chi cot\theta\partial_\theta + csc^2\chi\partial_\theta^2 + csc^2\chi csc^2\theta\partial_\phi^2]$

So, in Robertson-Walker coordinates on $\mathbf{R}^+ \times S^3$, the wave equation operator is,

$\frac{\partial^2}{\partial t^2} + 6R^{-1}\dot{R}\partial_t - \nabla_{S^3}^2$

$= \frac{\partial^2}{\partial t^2} + 6R^{-1}\dot{R}\partial_t - \frac{1}{R^2}[2cot\chi\partial_\chi + \partial_\chi^2 + csc\chi cot\theta\partial_\theta + csc^2\chi\partial_\theta^2 + csc^2\chi csc^2\theta\partial_\phi^2]$

The above equation is satisfied if both of the following equations are solved simultaneously:

$$\frac{\partial^2}{\partial t^2}\psi = \frac{1}{R^2}[\partial_\chi^2 + csc^2\chi\partial_\theta^2 + csc^2\chi csc^2\theta\partial_\phi^2]\psi \tag{9}$$

$$6R^{-1}\dot{R}\partial_t\psi = \frac{1}{R^2}[2cot\chi\partial_\chi + csc\chi cot\theta\partial_\theta]\psi \tag{10}$$

To solve (9) set $\psi = e^{Et+\Xi(\chi)+\Theta(\theta)+\Phi(\phi)}$

Then $E^2\psi = \frac{1}{R^2}[(\Xi'' + \Xi'^2) + csc^2\chi(\Theta'' + \Theta'^2) + csc^2\chi csc^2\theta(\Phi'' + \Phi'^2)]\psi$

and $E^2\psi - \frac{1}{R^2}[(\Xi'' + \Xi'^2) + csc^2\chi(\Theta'' + \Theta'^2) + csc^2\chi csc^2\theta(\Phi'' + \Phi'^2)]\psi = 0$

So, $[E^2 + g^{\chi\chi}(\Xi'' + \Xi'^2) + g^{\theta\theta}(\Theta'' + \Theta'^2) + g^{\phi\phi}(\Phi'' + \Phi'^2)]\psi = 0$

while at the same time $(E^2 + g_{\chi\chi}P_\chi^2 + g_{\theta\theta}P_\theta^2 + g_{\phi\phi}P_\phi^2)\psi = 0$

Setting $\Xi'' + \Xi'^2 = g^2_{\chi\chi}P^2_\chi$, $\Theta'' + \Theta'^2 = g^2_{\theta\theta}P^2_\theta$, $\Phi'' + \Phi'^2 = g^2_{\phi\phi}P^2_\phi$

We have Ξ, Θ, and Φ as the solutions to the following differential equations:
$\Xi'' + \Xi'^2 - g^2_{\chi\chi}P^2_\chi = 0$; $\Theta'' + \Theta'^2 - g^2_{\theta\theta}P^2_\theta = 0$; $\Phi'' + \Phi'^2 - g^2_{\phi\phi}P^2_\phi = 0$

In the special case where energy and momentum are constant the second derivatives are zero so we have the solutions for (9):

$$\psi = e^{\pm i(Et - R^2 P_\chi \chi - R^2 sin^2\chi P_\theta \theta - R^2 sin^2\chi sin^2\theta P_\phi \phi)}$$

From (10) we get:

$HE = \frac{1}{6R^2}[2cot\chi\Xi' + csc\chi cot\theta\Theta']$ where H is the Hubble parameter.

If both (9) and (10) are satisfied then the mixed equation will be satisfied but it is possible for a solution to satisfy the mixed equation without satisfying either (9) or (10). Such a situation could indicate the presence of 'exotic' matter.

To determine the wave equation in Schwarzschild coordinates we again use the 4D Laplacian.

$\nabla^2 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$.

For $\mathbf{R}^+ \times \mathbf{R}^+ \times S^2$,

$\nabla^2 = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta) = \frac{1}{\sqrt{-g}}[\partial_t(\sqrt{-g}g^{tt}\partial_t) + \partial_R(\sqrt{-g}g^{RR}\partial_R) + \partial_\theta(\sqrt{-g}g^{\theta\theta}\partial_\theta) + \partial_\phi(\sqrt{-g}g^{\phi\phi}\partial_\phi)]$

where $g_{\alpha\beta} = \begin{pmatrix} \kappa_R & 0 & 0 & 0 \\ 0 & -\kappa_R^{-1} & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2 sin^2\theta \end{pmatrix}$
and $g = Det(g_{\alpha\beta}) = -R^4 sin^2\theta$

and $\sqrt{-g} = R^2 sin\theta$

Then

$$\nabla^2 = \frac{1}{\sqrt{-g}}[\partial_t(\sqrt{-g})g^{tt}\partial_t + \partial_t(g^{tt})\sqrt{-g}\partial_t + \sqrt{-g}g^{tt}\partial_t^2$$

$$+\partial_R(\sqrt{-g})g^{RR}\partial_R + \partial_R(g^{RR})\sqrt{-g}\partial_R + \sqrt{-g}g^{RR}\partial_R^2$$

$$+\partial_\theta(\sqrt{-g})g^{\theta\theta}\partial_\theta + \partial_\theta(g^{\theta\theta})\sqrt{-g}\partial_\theta + \sqrt{-g}g^{\theta\theta}\partial_\theta^2$$

$$+\partial_\phi(\sqrt{-g})g^{\phi\phi}\partial_\phi + \partial_\phi(g^{\phi\phi})\sqrt{-g}\partial_\phi + \sqrt{-g}g^{\phi\phi}\partial_\phi^2]$$

Then

$$\nabla^2 = \frac{1}{\sqrt{-g}}[\frac{R^2 sin\theta}{\kappa_R}\partial_t^2 + \partial_R(\sqrt{-g})g^{RR}\partial_R + \partial_R(g^{RR})\sqrt{-g}\partial_R + \sqrt{-g}g^{RR}\partial_R^2$$

$$+\partial_\theta(\sqrt{-g})g^{\theta\theta}\partial_\theta + \partial_\theta(g^{\theta\theta})\sqrt{-g}\partial_\theta + \sqrt{-g}g^{\theta\theta}\partial_\theta^2$$

$$+\partial_\phi(\sqrt{-g})g^{\phi\phi}\partial_\phi + \partial_\phi(g^{\phi\phi})\sqrt{-g}\partial_\phi + \sqrt{-g}g^{\phi\phi}\partial_\phi^2]$$

Then

$$\nabla^2 = \kappa_R^{-1}\partial_t^2 + \frac{1}{\sqrt{-g}}[\partial_R(\sqrt{-g})g^{RR}\partial_R + \partial_R(g^{RR})\sqrt{-g}\partial_R + \sqrt{-g}g^{RR}\partial_R^2$$

$$+\partial_\theta(\sqrt{-g})g^{\theta\theta}\partial_\theta + \partial_\theta(g^{\theta\theta})\sqrt{-g}\partial_\theta + \sqrt{-g}g^{\theta\theta}\partial_\theta^2$$

$$+\partial_\phi(\sqrt{-g})g^{\phi\phi}\partial_\phi + \partial_\phi(g^{\phi\phi})\sqrt{-g}\partial_\phi + \sqrt{-g}g^{\phi\phi}\partial_\phi^2]$$

$$= \kappa_R^{-1}\partial_t^2 - \frac{1}{\sqrt{\eta}}[\partial_R(\sqrt{\eta})\eta^{RR}\partial_R + \partial_R(\eta^{RR})\sqrt{\eta}\partial_R + \sqrt{\eta}\eta^{RR}\partial_R^2$$

$$+\partial_\theta(\sqrt{\eta})\eta^{\theta\theta}\partial_\theta + \partial_\theta(\eta^{\theta\theta})\sqrt{\eta}\partial_\theta + \sqrt{\eta}\eta^{\theta\theta}\partial_\theta^2$$

$$+\partial_\phi(\sqrt{\eta})\eta^{\phi\phi}\partial_\phi + \partial_\phi(\eta^{\phi\phi})\sqrt{\eta}\partial_\phi + \sqrt{\eta}\eta^{\phi\phi}\partial_\phi^2]$$

where $\eta_{ij} = \begin{pmatrix} \kappa_R^{-1} & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 sin^2\theta \end{pmatrix}$

and $\eta = Det(\eta_{ij}) = \kappa_R^{-1} R^4 sin^2\theta$

and $\sqrt{\eta} = \kappa_R^{-1/2} R^2 sin\theta$

We set $\nabla^2_{\mathbf{R}^+ \times S^2} = \frac{1}{\sqrt{\eta}}[\partial_R(\sqrt{\eta})\eta^{RR}\partial_R + \partial_R(\eta^{RR})\sqrt{\eta}\partial_R + \sqrt{\eta}\eta^{RR}\partial_R^2$

$+\partial_\theta(\sqrt{\eta})\eta^{\theta\theta}\partial_\theta + \partial_\theta(\eta^{\theta\theta})\sqrt{\eta}\partial_\theta + \sqrt{\eta}\eta^{\theta\theta}\partial_\theta^2$

$+\partial_\phi(\sqrt{\eta})\eta^{\phi\phi}\partial_\phi + \partial_\phi(\eta^{\phi\phi})\sqrt{\eta}\partial_\phi + \sqrt{\eta}\eta^{\phi\phi}\partial_\phi^2]$

$= \frac{1}{\sqrt{\eta}}[\partial_R(\sqrt{\eta}\eta^{RR})\partial_R + \sqrt{\eta}\eta^{RR}\partial_R^2$

$+\partial_\theta(\sqrt{\eta}\eta^{\theta\theta})\partial_\theta + \sqrt{\eta}\eta^{\theta\theta}\partial_\theta^2$

$+\partial_\phi(\sqrt{\eta}\eta^{\phi\phi})\partial_\phi + \sqrt{\eta}\eta^{\phi\phi}\partial_\phi^2]$

$= \frac{1}{\sqrt{\eta}}[\partial_R(R^2 sin\theta)\partial_R + R^2 sin\theta\partial_R^2$

$+\partial_\theta(\kappa_R^{-1/2} sin\theta)\partial_\theta + \kappa_R^{-1/2} sin\theta\partial_\theta^2$

$+\partial_\phi(\kappa_R^{-1/2})\partial_\phi + \kappa_R^{-1/2}\partial_\phi^2]$

$= \frac{1}{\sqrt{\eta}}[2R sin\theta\partial_R + R^2 sin\theta\partial_R^2 + \kappa_R^{-1/2} cos\theta\partial_\theta + \kappa_R^{-1/2} sin\theta\partial_\theta^2 + \kappa_R^{-1/2}\partial_\phi^2]$

$= \frac{1}{R^2}[2R\kappa_R^{1/2}\partial_R + R^2\kappa_R^{1/2}\partial_R^2 + cot\theta\partial_\theta + \partial_\theta^2 + csc\theta\partial_\phi^2]$

Then the wave equation operator in Schwarzschild coordinates is

$\kappa_R^{-1}\partial_t^2 - \nabla^2_{\mathbf{R}^+\times S^2}$

$= \kappa_R^{-1}\partial_t^2 - \frac{1}{R^2}[2R\kappa_R^{1/2}\partial_R + R^2\kappa_R^{1/2}\partial_R^2 + cot\theta\partial_\theta + \partial_\theta^2 + csc\theta\partial_\phi^2]$

In the weak field limit where $\kappa_R \approx 1$, $\nabla^2_{\mathbf{R}^+\times S^2}$ reduces to the Laplacian in ordinary polar coordinates. So, in the weak field limit the operator is as mentioned above, $\frac{\partial^2}{\partial t^2} - \nabla^2$.

Referring back to the article *The Z-field (Energy-Momentum)* we can now summarize. We will call Type 1 equations the set $\{(9),(10),(12),(14)\}$ in that article and Type 2 the set $\{(9),(11),(13),(15)\}$.

Bosons:–Satisfy Type 1 with wavelength λ– Satisfy Klein-Gordon Eqn with $p = \frac{hc}{\lambda}$

Fermions:–Satisfy Type 2 with wavelength λ – Satisfy Dirac eqn with $p = \frac{hc}{\lambda}$

Footnote*: These matrices are constructed as follows:

Start with the quaternions in $M_{4\times 4}\{0,\pm 1\}$

$$\mathbf{1}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\ \mathbf{i}=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$$
$$\mathbf{j}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}\ \mathbf{k}=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}.$$

Then form the matrices $\boldsymbol{i}\mathbf{i}, \boldsymbol{i}\mathbf{j}, \boldsymbol{i}\mathbf{k}$

$$\text{where } \boldsymbol{i}=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$$

Then

$$(E_0,E_1,E_2,E_3,E_4,E_5,E_6,E_7)=(\mathbf{1},\boldsymbol{i},\mathbf{i},\boldsymbol{i}\mathbf{i},\mathbf{j},\boldsymbol{i}\mathbf{j},\mathbf{k},\boldsymbol{i}\mathbf{k})$$

As stated previously, their multiplication table is

$\times$	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_0	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_1	E_1	$-E_0$	E_3	$-E_2$	E_5	$-E_4$	E_7	$-E_6$
E_2	E_2	E_3	$-E_0$	$-E_1$	E_6	E_7	$-E_4$	$-E_5$
E_3	E_3	$-E_2$	$-E_1$	E_0	E_7	$-E_6$	$-E_5$	E_4
E_4	E_4	E_5	$-E_6$	$-E_7$	$-E_0$	$-E_1$	E_2	E_3
E_5	E_5	$-E_4$	$-E_7$	E_6	$-E_1$	E_0	E_3	$-E_2$
E_6	E_6	E_7	E_4	E_5	$-E_2$	$-E_3$	$-E_0$	$-E_1$
E_7	E_7	$-E_6$	E_5	$-E_4$	$-E_3$	E_2	$-E_1$	E_0

Octonians:

The following is the Cayley-Graves multiplication table for the octonians.(http://en.wikipedia.org/wiki/Octonion)

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$(-e_7)$	(e_6)
e_2	e_2	$(-e_3)$	$-e_0$	(e_1)	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	(e_2)	$-e_1$	$(-e_0)$	e_7	$-e_6$	(e_5)	$(-e_4)$
e_4	e_4	$(-e_5)$	$-e_6$	$-e_7$	$-e_0$	(e_1)	e_2	e_3
e_5	e_5	(e_4)	$-e_7$	e_6	$-e_1$	$(-e_0)$	$(-e_3)$	(e_2)
e_6	e_6	e_7	e_4	$(-e_5)$	$-e_2$	(e_3)	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	(e_4)	$-e_3$	$(-e_2)$	(e_1)	$(-e_0)$

Comparing this table to that for the Es derived above we can make the association $e_i \leftrightarrow E_i$. Then $e_ie_j \leftrightarrow E_iE_j$ except for the entries in parentheses in which case $e_ie_j \leftrightarrow -E_iE_j$.

We observe a similar structure for the odd octonians

$\times$	e_0	e_3	e_5	e_7
e_0	e_0	e_3	e_5	e_7
e_3	e_3	$(-e_0)$	$-e_1e_7$	e_1e_5
e_5	e_5	e_1e_7	$(-e_0)$	$-e_1e_3$
e_7	e_7	$-e_1e_5$	e_1e_3	$(-e_0)$

as for the Pauli matrices

$\times$	I	σ_1	σ_2	σ_3
I	I	σ_1	σ_2	σ_3
σ_1	σ_1	I	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	I	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	I

Let $\phi(\pm\sigma_1, \pm\sigma_2, \pm\sigma_3) = (\pm e_3, \pm e_5, \pm e_7)$. Then $e_1\phi(xy/i) = -\phi(x)\phi(y)$.

Footnote**:

The solution

$$\phi = [\tfrac{1}{E+m}\begin{pmatrix} \hat{p}_z & 0 \\ 0 & \hat{p}_z \\ \hat{p}_x & -\hat{p}_y \\ \hat{p}_y & \hat{p}_x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}]e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$$

to the Dirac equation involves a slight abuse of notation.

Let $\theta = Et - \mathbf{p}\cdot\mathbf{r}$

$e^{-E_1\theta}$ is the 4×4 matrix

$$e^{-E_1\theta} = \begin{pmatrix} cos\theta & sin\theta & 0 & 0 \\ -sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \\ 0 & 0 & -sin\theta & cos\theta \end{pmatrix} = cos\theta E_0 - sin\theta E_1.$$

$\begin{pmatrix} \hat{p}_z & 0 \\ 0 & \hat{p}_z \\ \hat{p}_x & -\hat{p}_y \\ \hat{p}_y & \hat{p}_x \end{pmatrix} e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}$ is to be understood to mean

$$\begin{pmatrix} \hat{p}_z(cos\theta E_0 - sin\theta E_1) & 0 \\ 0 & \hat{p}_z(cos\theta E_0 - sin\theta E_1) \\ \hat{p}_x(cos\theta E_0 - sin\theta E_1) & -\hat{p}_y(cos\theta E_0 - sin\theta E_1) \\ \hat{p}_y(cos\theta E_0 - sin\theta E_1) & \hat{p}_x(cos\theta E_0 - sin\theta E_1) \end{pmatrix}$$

$$= -E_1 \begin{pmatrix} \partial_z(cos\theta E_0 - sin\theta E_1) & 0 \\ 0 & \partial_z(cos\theta E_0 - sin\theta E_1) \\ \partial_x(cos\theta E_0 - sin\theta E_1) & -\partial_y(cos\theta E_0 - sin\theta E_1) \\ \partial_y(cos\theta E_0 - sin\theta E_1) & \partial_x(cos\theta E_0 - sin\theta E_1) \end{pmatrix}$$

$$= -E_1 \begin{pmatrix} -p_z(-sin\theta E_0 - cos\theta E_1) & 0 \\ 0 & -p_z(-sin\theta E_0 - cos\theta E_1) \\ -p_x(-sin\theta E_0 - cos\theta E_1) & p_y(-sin\theta E_0 - cos\theta E_1) \\ -p_y(-sin\theta E_0 - cos\theta E_1) & -p_x(-sin\theta E_0 - cos\theta E_1) \end{pmatrix}$$

$$= -E_1 \begin{pmatrix} p_z(sin\theta E_0 + cos\theta E_1) & 0 \\ 0 & p_z(sin\theta E_0 + cos\theta E_1) \\ p_x(sin\theta E_0 + cos\theta E_1) & -p_y(sin\theta E_0 + cos\theta E_1) \\ p_y(sin\theta E_0 + cos\theta E_1) & p_x(sin\theta E_0 + cos\theta E_1) \end{pmatrix}$$

$$= \begin{pmatrix} p_z(cos\theta E_0 - sin\theta E_1) & 0 \\ 0 & p_z(cos\theta E_0 - sin\theta E_1) \\ p_x(cos\theta E_0 - sin\theta E_1) & -p_y(cos\theta E_0 - sin\theta E_1) \\ p_y(cos\theta E_0 - sin\theta E_1) & p_x(cos\theta E_0 - sin\theta E_1) \end{pmatrix}$$

$$= \begin{pmatrix} p_z & 0 \\ 0 & p_z \\ p_x & -p_y \\ p_y & p_x \end{pmatrix} e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})}.$$

It should also be noted that there is a diffeomorphism

$$\begin{pmatrix} cos\theta & sin\theta & 0 & 0 \\ -sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \\ 0 & 0 & -sin\theta & cos\theta \end{pmatrix} \leftrightarrow \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

indicating that θ traces out the equivalent of a circular path around the torus

$$\begin{pmatrix} cos\theta & sin\theta & 0 & 0 \\ -sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & cos\phi & sin\phi \\ 0 & 0 & -sin\phi & cos\phi \end{pmatrix}$$

So, in fact $e^{-E_1(Et-\mathbf{p}\cdot\mathbf{r})} \simeq e^{-i(Et-\mathbf{p}\cdot\mathbf{r})}$

and our solution can be written as

$$\phi = [\frac{1}{E+m}\begin{pmatrix} \hat{p}_z & 0 \\ 0 & \hat{p}_z \\ \hat{p}_x & -\hat{p}_y \\ \hat{p}_y & \hat{p}_x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}]e^{-i(Et-\mathbf{p}\cdot\mathbf{r})}$$

or as

$$\phi = [\frac{1}{E+m}\begin{pmatrix} \hat{p}_z \\ \hat{p}_x + i\hat{p}_y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}]e^{-i(Et-\mathbf{p}\cdot\mathbf{r})}$$

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