Intro: Quaternions and biQuaternions

The space of quaternions is the span (set of all linear combinations over $\mathbf{R}$), denoted by $\mathbf{H}$, of the matrices:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \mathbf{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Their multiplication table is:

$\times$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

We refer to $\mathbf{1}$ as the real basis member and the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as the imaginary basis members of $\mathbf{H}$, Quaternion space. It can easily be shown directly that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ anti-commute and that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$ and $\mathbf{ijk} = -\mathbf{1}$. Given a quaternion $\mathbf{q} = w\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we denote by $\mathbf{q}^* = w\mathbf{1} - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$ and the norm $||\cdot||$ is defined by $||\mathbf{q}|| = \sqrt{\mathbf{qq}^*}$.

We could also form the span over all complex numbers and get the bi-Quaternions $span_{\mathbf{C}}(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) = span_{\mathbf{R}}(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) \oplus span_{i\mathbf{R}}(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ where $i\mathbf{R}$ is the set of pure imaginary numbers in $\mathbf{C}$. We note that $span_{i\mathbf{R}}(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) = span_{\mathbf{R}}(i\mathbf{1}, i\mathbf{i}, i\mathbf{j}, i\mathbf{k})$.

We have already encountered the biQuaternions in previous articles as the span of $\mathbf{E}$ where a basis for $\mathbf{E}$ was given as:

$$E_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} E_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad E_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Their multiplication table is

$\times$	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_0	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_1	E_1	$-E_0$	E_3	$-E_2$	E_5	$-E_4$	E_7	$-E_6$
E_2	E_2	E_3	$-E_0$	$-E_1$	E_6	E_7	$-E_4$	$-E_5$
E_3	E_3	$-E_2$	$-E_1$	E_0	E_7	$-E_6$	$-E_5$	E_4
E_4	E_4	E_5	$-E_6$	$-E_7$	$-E_0$	$-E_1$	E_2	E_3
E_5	E_5	$-E_4$	$-E_7$	E_6	$-E_1$	E_0	E_3	$-E_2$
E_6	E_6	E_7	E_4	E_5	$-E_2$	$-E_3$	$-E_0$	$-E_1$
E_7	E_7	$-E_6$	E_5	$-E_4$	$-E_3$	E_2	$-E_1$	E_0

We can form the correspondence

$$\{E_0, E_2, E_4, E_6\} \sim \{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\} \text{ and } \{E_1, E_3, E_5, E_7\} \sim \{i\mathbf{1}, i\mathbf{i}, i\mathbf{j}, i\mathbf{k}\}$$

We can also form the correspondence $\{E_3, E_5, E_7\} \sim \{\sigma_1, \sigma_2, \sigma_3\}$ where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

whose multiplication table is:

$\times$	I	σ_1	σ_2	σ_3
I	I	σ_1	σ_2	σ_3
σ_1	σ_1	I	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	I	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	I

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