Quaternion Space-Time

In Special Relativity the metric $ds^2 = cdt^2 - dx^2 - dy^2 - dz^2$ is invariant under inertial transformations. For convenience we adopt units where $c = 1$. Such invariance means observers with constant relative velocity will measure ds^2 to be the same.

For example, an event with coordinate displacement $(\Delta t_1, \Delta x_1, \Delta y_1, \Delta z_1)$ with respect to observer 1 and $(\Delta t_2, \Delta x_2, \Delta y_2, \Delta z_2)$ with respect to observer 2 has the property that $\Delta t_1^2 - \Delta x_1^2 - \Delta y_1^2 - \Delta z_1^2 = \Delta t_2^2 - \Delta x_2^2 - \Delta y_2^2 - \Delta z_2^2$.

Knowing the coordinate displacement with respect to one observer we can find the corresponding displacement with respect to the other observer using the Lorentz transformation.

The Lorentz transformation describes space-time transformation between inertial frames in motion relative to each other.* We assume for simplicity that the direction of motion is along the x-axis for each. Then the coordinates transform according to:

$$\begin{pmatrix} \Delta t_2 \\ \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t_1 \\ \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \end{pmatrix}$$

where $cosh(\alpha) = \frac{1}{\sqrt{1-v^2}}$ and $sinh(\alpha) = \frac{v}{\sqrt{1-v^2}}$. $cosh(\alpha)$ is always positive but $sinh(\alpha)$ can be positive or negative depending on the direction of motion.

The above metric $ds^2 = cdt^2 - dx^2 - dy^2 - dz^2$ is referred to as the Minkowski metric. It has a very natural expression with respect to quaternions.

Let $\mathbf{H} = span\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \mathbf{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

It can easily be shown that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ anti-commute and that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$ and $\mathbf{ijk} = -\mathbf{1}$.

The quaternion multiplication table is:

$\times$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

The space of quaternions ($\mathbf{H} = span\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ over $\mathbf{R}$) forms a division algebra with an associated exponential $exp(\mathbf{H}) = \mathbf{R}^+ \times S^3$.

$exp(\tau\mathbf{1} + \chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k}) = e^\tau exp(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})$

Want to find $exp(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k}) = \Sigma_{n=0}^{\infty} \frac{(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^n}{n!}$

$(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})^0 = 1$

$(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})^1 = \chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k}$

$(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})^2 = -\chi^2 - \theta^2 - \phi^2 = -(\chi^2 + \theta^2 + \phi^2)$

$(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})^3 = -(\chi^2 + \theta^2 + \phi^2)(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})$

$= -\chi(\chi^2 + \theta^2 + \phi^2)\mathbf{i} + -\theta(\chi^2 + \theta^2 + \phi^2)\mathbf{j} - \phi(\chi^2 + \theta^2 + \phi^2)\mathbf{k}$

$(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})^4 = (-\chi(\chi^2 + \theta^2 + \phi^2)\mathbf{i} - \theta(\chi^2 + \theta^2 + \phi^2)\mathbf{j} - \phi(\chi^2 + \theta^2 + \phi^2)\mathbf{k})(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})$

$= +(\chi^2 + \theta^2 + \phi^2)^2$

$(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})^5 = +(\chi^2 + \theta^2 + \phi^2)^2(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})$

$= \chi(\chi^2+\theta^2+\phi^2)^2\mathbf{i}+\theta(\chi^2+\theta^2+\phi^2)^2\mathbf{j}+\phi(\chi^2+\theta^2+\phi^2)^2\mathbf{k}$

$(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^6 = (\chi(\chi^2+\theta^2+\phi^2)^2\mathbf{i}+\theta(\chi^2+\theta^2+\phi^2)^2\mathbf{j}+\phi(\chi^2+\theta^2+\phi^2)^2\mathbf{k})(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})$

$= -\chi^2(\chi^2+\theta^2+\phi^2)^2-\theta^2(\chi^2+\theta^2+\phi^2)^2-\phi^2(\chi^2+\theta^2+\phi^2)^2$

$= -(\chi^2+\theta^2+\phi^2)^3$

$(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^7 = -\chi(\chi^2+\theta^2+\phi^2)^3\mathbf{i}-\theta(\chi^2+\theta^2+\phi^2)^3\mathbf{j}-\phi(\chi^2+\theta^2+\phi^2)^3\mathbf{k}$

$(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^8 = (-\chi(\chi^2+\theta^2+\phi^2)^3\mathbf{i}-\theta(\chi^2+\theta^2+\phi^2)^3\mathbf{j}-\phi(\chi^2+\theta^2+\phi^2)^3\mathbf{k})(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})$

$= +\chi^2(\chi^2+\theta^2+\phi^2)^3+\theta^2(\chi^2+\theta^2+\phi^2)^3+\chi^2(\chi^2+\theta^2+\phi^2)^3$

$= +(\chi^2+\theta^2+\phi^2)^4$

$(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^9 = \chi(\chi^2+\theta^2+\phi^2)^4\mathbf{i}+\theta(\chi^2+\theta^2+\phi^2)^4\mathbf{j}+\phi(\chi^2+\theta^2+\phi^2)^4\mathbf{k}$

The pattern is:

$(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^{2n} = (-1)^n(\chi^2+\theta^2+\phi^2)^n$

$(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^{2n+1}$

$= (-1)^n[\chi(\chi^2+\theta^2+\phi^2)^n\mathbf{i}+\theta(\chi^2+\theta^2+\phi^2)^n\mathbf{j}+\phi(\chi^2+\theta^2+\phi^2)^n\mathbf{k}]$

$exp(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k}) = \Sigma_{n=0}^{\infty}\frac{(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^n}{n!}$

$= \Sigma_{n=0}^{\infty}\frac{(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^{2n}}{(2n)!}+\Sigma_{n=0}^{\infty}\frac{(\chi\mathbf{i}+\theta\mathbf{j}+\phi\mathbf{k})^{2n+1}}{(2n+1)!}$

$= \Sigma_{n=0}^{\infty}(-1)^n\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n)!}$

$+\chi\Sigma_{n=0}^{\infty}(-1)^n\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}\mathbf{i}+\theta\Sigma_{n=0}^{\infty}(-1)^n\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}\mathbf{j}+\phi\Sigma_{n=0}^{\infty}(-1)^n\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}\mathbf{k}$

Let $\alpha = \Sigma_{n=0}^{\infty}(-1)^n\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n)!}$ and $\beta = \Sigma_{n=0}^{\infty}(-1)^n\frac{(\chi^2+\theta^2+\phi^2)^n}{(2n+1)!}$

Then $exp\begin{pmatrix} 0 & -\chi & \theta & -\phi \\ \chi & 0 & \phi & \theta \\ -\theta & -\phi & 0 & \chi \\ \phi & -\theta & -\chi & 0 \end{pmatrix} = \begin{pmatrix} \alpha & -\chi\beta & \theta\beta & -\phi\beta \\ \chi\beta & \alpha & \phi\beta & \theta\beta \\ -\theta\beta & -\phi\beta & \alpha & \chi\beta \\ \phi\beta & -\theta\beta & -\chi\beta & \alpha \end{pmatrix}$

$= \alpha I + \beta \begin{pmatrix} 0 & -\chi & \theta & -\phi \\ \chi & 0 & \phi & \theta \\ -\theta & -\phi & 0 & \chi \\ \phi & -\theta & -\chi & 0 \end{pmatrix} = \alpha\mathbf{1} + \beta(\chi\mathbf{i} + \theta\mathbf{j} + \phi\mathbf{k})$

For the case $\chi \neq 0, \theta = 0, \phi = 0$

$\alpha = \Sigma_{n=0}^{\infty}(-1)^n \frac{(\chi^2)^n}{(2n)!} = \Sigma_{n=0}^{\infty}(-1)^n \frac{\chi^{2n}}{(2n)!} = cos\chi$

and $\chi\beta = \chi\Sigma_{n=0}^{\infty}(-1)^n \frac{(\chi^2)^n}{(2n+1)!} = \Sigma_{n=0}^{\infty}(-1)^n \frac{\chi^{2n+1}}{(2n+1)!} = sin\chi$

Then $exp\begin{pmatrix} 0 & -\chi & 0 & 0 \\ \chi & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi \\ 0 & 0 & -\chi & 0 \end{pmatrix} = \begin{pmatrix} cos\chi & -sin\chi & 0 & 0 \\ sin\chi & cos\chi & 0 & 0 \\ 0 & 0 & cos\chi & sin\chi \\ 0 & 0 & -sin\chi & cos\chi \end{pmatrix}$

$= cos\chi\mathbf{1} + sin\chi\mathbf{i}$

For a quaternion $\mathbf{q} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, $\|\mathbf{q}\| = \sqrt{\mathbf{q}\mathbf{q}^*}$

where $\mathbf{q}^* = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$

Then $\|cos\chi\mathbf{1} + sin\chi\mathbf{i}\| = \sqrt{cos^2\chi + sin^2\chi} = 1$

For the case $\chi = 0, \theta \neq 0, \phi = 0$

$\alpha = \Sigma_{n=0}^{\infty}(-1)^n \frac{(\theta^2)^n}{(2n)!} = \Sigma_{n=0}^{\infty}(-1)^n \frac{\theta^{2n}}{(2n)!} = cos\theta$

and $\theta\beta = \theta\Sigma_{n=0}^{\infty}(-1)^n \frac{(\theta^2)^n}{(2n+1)!} = \Sigma_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} = sin\theta$

Then $exp\begin{pmatrix} 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \theta \\ -\theta & 0 & 0 & 0 \\ 0 & -\theta & 0 & 0 \end{pmatrix} = \begin{pmatrix} cos\theta & 0 & sin\theta & 0 \\ 0 & cos\theta & 0 & sin\theta \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & -sin\theta & 0 & cos\theta \end{pmatrix}$

$= cos\theta\mathbf{1} + sin\theta\mathbf{j}$

$\|cos\theta\mathbf{1} + sin\theta\mathbf{j}\| = \sqrt{cos^2\theta + sin^2\theta} = 1$

For the case $\chi = 0, \theta = 0, \phi \neq 0$

$\alpha = \Sigma_{n=0}^{\infty}(-1)^n \frac{(\phi^2)^n}{(2n)!} = \Sigma_{n=0}^{\infty}(-1)^n \frac{\phi^{2n}}{(2n)!} = cos\phi$

and $\phi\beta = \phi\Sigma_{n=0}^{\infty}(-1)^n \frac{(\phi^2)^n}{(2n+1)!} = \Sigma_{n=0}^{\infty}(-1)^n \frac{\phi^{2n+1}}{(2n+1)!} = sin\phi$

$$exp\begin{pmatrix} 0 & 0 & 0 & -\phi \\ 0 & 0 & \phi & 0 \\ 0 & -\phi & 0 & 0 \\ \phi & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} cos\phi & 0 & 0 & -sin\phi \\ 0 & cos\phi & sin\phi & 0 \\ 0 & -sin\phi & cos\phi & 0 \\ sin\phi & 0 & 0 & cos\phi \end{pmatrix}$$

$= cos\phi\mathbf{1} + sin\phi\mathbf{k}$

$\|cos\phi\mathbf{1} + sin\phi\mathbf{k}\| = \sqrt{cos^2\phi + sin^2\phi} = 1$

$$exp(\mathbf{H}) = \{e^{\tau} exp\begin{pmatrix} 0 & -\chi & \theta & -\phi \\ \chi & 0 & \phi & \theta \\ -\theta & -\phi & 0 & \chi \\ \phi & -\theta & -\chi & 0 \end{pmatrix} : (\tau, \chi, \theta, \phi) \in \mathbf{R}^3\}$$

and $exp(Im\mathbf{H}) \cong S^3$ since

$\{\mathbf{q} : ||\mathbf{q}|| = \sqrt{\alpha^2 + \beta^2\chi^2 + \beta^2\theta^2 + \beta^2\phi^2} = 1\} \cong S^3$.

So, $exp(\mathbf{H}) = \mathbf{R}^+ \times S^3$.

The Lie group SU(2) given by $\{a\mathbf{1} + b\mathbf{i} - c\mathbf{j} + d\mathbf{k} : a^2 + b^2 + c^2 + d^2 = 1\}$ is diffeomorphic to $S^3 = \{a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a^2 + b^2 + c^2 + d^2 = 1\}$ and has as generators the set $\{\mathbf{i}, -\mathbf{j}, \mathbf{k}\}$. SU(2) is used in the description of electroweak interactions and beta decay.

The Lie algebra su(2)=span$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ has the commutation relations

$[\mathbf{i}, \mathbf{j}] = 2\mathbf{k}$, $[\mathbf{j}, \mathbf{k}] = 2\mathbf{i}$, and $[\mathbf{k}, \mathbf{i}] = 2\mathbf{j}$.

Given two quaternions $\mathbf{q} = q_1\mathbf{1} + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$

and $\mathbf{r} = r_1\mathbf{1} + r_2\mathbf{i} + r_3\mathbf{j} + r_4\mathbf{k}$, it is easy to show that

$\mathbf{qr} = q_1\mathbf{r} + r_1\mathbf{q} + Im\mathbf{q} \times Im\mathbf{r} - Im\mathbf{q} \cdot Im\mathbf{r}$ where '$\times$' and '$\cdot$' are the standard vector operations.

If both $\mathbf{q}$ and $\mathbf{r}$ are pure imaginary then

$\mathbf{qr} = \mathbf{q} \times \mathbf{r} - \mathbf{q} \cdot \mathbf{r}$

The Maxwell equations in vacuum are:

$\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and

$\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$

where $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$.

The first pair can be expressed in quaternion form as:

$\nabla \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and the second as $\nabla \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$.

In units where $c = 1$, Maxwell's equations can be expressed in the form

$$\begin{pmatrix} \nabla \mathbf{E} \\ \nabla \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial \mathbf{E}/\partial t \\ \partial \mathbf{B}/\partial t \end{pmatrix}$$

Returning to the Minkowski metric, it is expressed in quaternion form as

$$d\mathbf{S}^2 = (dt\mathbf{1})^2 + (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})^2 = dt^2\mathbf{1} - dx^2\mathbf{1} - dy^2\mathbf{1} - dz^2\mathbf{1}.$$

which can also be expressed as

$$d\mathbf{S}^2 = \begin{pmatrix} dt & dx & dy & dz \end{pmatrix} \begin{pmatrix} \mathbf{11} & 0 & 0 & 0 \\ 0 & \mathbf{ii} & 0 & 0 \\ 0 & 0 & \mathbf{jj} & 0 \\ 0 & 0 & 0 & \mathbf{kk} \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

Setting $*dt\mathbf{1} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ we can then also express the Minkowski metric as $d\mathbf{S}^2 = (dt\mathbf{1})^2 + (*dt\mathbf{1})^2$.

We can identify a real number u with $u\mathbf{1}$ and express this as $u \simeq u\mathbf{1}$. Then $ds^2 \simeq d\mathbf{S}^2$.

Let M be a real symmetric bilinear form. It can be diagonalized by some invertible Q where $D = QMQ^{-1}$.

We can say that M *preserves quaternion structure* if for co-ordinates t, u, v, w and quaternion frame $\{\mathbf{1}, \mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$,

$QMQ^{-1}\mathbf{1}$

$$= \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{e}_u & 0 & 0 \\ 0 & 0 & \mathbf{e}_v & 0 \\ 0 & 0 & 0 & \mathbf{e}_w \end{pmatrix} \begin{pmatrix} |D_{00}| & 0 & 0 & 0 \\ 0 & |D_{11}| & 0 & 0 \\ 0 & 0 & |D_{22}| & 0 \\ 0 & 0 & 0 & |D_{33}| \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{e}_u & 0 & 0 \\ 0 & 0 & \mathbf{e}_v & 0 \\ 0 & 0 & 0 & \mathbf{e}_w \end{pmatrix}$$

Preserving quaternion structure equates to preservation of the metric signature (+,-,-,-).

The motion of a mass-less point in a gravitational field satisfies the geodesic equation(s)

$$\frac{d^2x_a}{ds^2} = \Sigma_{\mu,\nu}\Gamma^{\alpha}_{\mu\nu}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}$$

where the Γs satisfy the field equation(s)

$$\Sigma_\alpha \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\alpha} + \Sigma_{\alpha,\beta}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} = 0 \text{ and } Det(g_{\mu\nu}) = -1, \text{ where } \Gamma^\alpha_{\mu\nu} = -\frac{1}{2}\Sigma_\beta g^{\alpha\beta}\left(\frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta}\right).$$

(See Schwarzschild[1916])

The condition that the determinant $Det(g_{\mu\nu}) = -1$, requires that a solution to the field equation(s) has a symmetric bi-linear representation $(g_{\mu\nu})$ with *determinant=-1.*

For the diagonal matrix above, this implies that $\Pi_\alpha |D_{\alpha\alpha}| = 1$.

Let $\{\mathbf{1}, \mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ be a quaternion basis with multiplication table,

$\times$	$\mathbf{1}$	$\mathbf{e}_u$	$\mathbf{e}_v$	$\mathbf{e}_w$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_u$	$\mathbf{e}_v$	$\mathbf{e}_w$
$\mathbf{e}_u$	$\mathbf{e}_u$	$-\mathbf{1}$	$\mathbf{e}_w$	$-\mathbf{e}_v$
$\mathbf{e}_v$	$\mathbf{e}_v$	$-\mathbf{e}_w$	$-\mathbf{1}$	$\mathbf{e}_u$
$\mathbf{e}_w$	$\mathbf{e}_w$	$\mathbf{e}_v$	$-\mathbf{e}_u$	$-\mathbf{1}$

Then $ds^2 \simeq d\mathbf{S}^2$

$$= \begin{pmatrix} dt & du & dv & dw \end{pmatrix} \begin{pmatrix} \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle \mathbf{11} & 0 & 0 & 0 \\ 0 & \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle \mathbf{e}_u \mathbf{e}_u & 0 & 0 \\ 0 & 0 & \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle \mathbf{e}_v \mathbf{e}_v & 0 \\ 0 & 0 & 0 & \langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle \mathbf{e}_w \mathbf{e}_w \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

$$= \begin{pmatrix} dt & du & dv & dw \end{pmatrix} \begin{pmatrix} ||\frac{\partial}{\partial t}||^2 \mathbf{11} & 0 & 0 & 0 \\ 0 & ||\frac{\partial}{\partial u}||^2 \mathbf{e}_u \mathbf{e}_u & 0 & 0 \\ 0 & 0 & ||\frac{\partial}{\partial v}||^2 \mathbf{e}_v \mathbf{e}_v & 0 \\ 0 & 0 & 0 & ||\frac{\partial}{\partial w}||^2 \mathbf{e}_w \mathbf{e}_w \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}$$

$$\simeq \begin{pmatrix} dt & du & dv & dw \end{pmatrix} \begin{pmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{uu} & 0 & 0 \\ 0 & 0 & g_{vv} & 0 \\ 0 & 0 & 0 & g_{ww} \end{pmatrix} \begin{pmatrix} dt \\ du \\ dv \\ dw \end{pmatrix}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric in $\mathbf{R}^4$ and g has signature (+,-,-,-).

We can describe $\mathbf{R}^4$ in polar coordinates and get a metric analogous to the Robertson-Walker metric for spherical coordinates:

Let

$$\begin{aligned} x &= R cos\chi \\ y &= R sin\chi cos\theta \\ z &= R sin\chi sin\theta cos\phi \\ \eta &= R sin\chi sin\theta sin\phi \end{aligned}$$

Using the chain rule:

$$\begin{aligned} \frac{\partial}{\partial R} &= \frac{\partial x}{\partial R}\frac{\partial}{\partial x} + \frac{\partial y}{\partial R}\frac{\partial}{\partial y} + \frac{\partial z}{\partial R}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial R}\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \chi} &= \frac{\partial x}{\partial \chi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \chi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \chi}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial \chi}\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial \theta}\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial \phi}\frac{\partial}{\partial \eta} \end{aligned}$$

Now, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \eta}$ are orthogonal unit vectors in $\mathbf{R}^4$ so

$$\begin{aligned} \langle \frac{\partial}{\partial R}, \frac{\partial}{\partial R} \rangle &= 1 \\ \langle \frac{\partial}{\partial \chi}, \frac{\partial}{\partial \chi} \rangle &= R^2 \\ \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle &= R^2 sin^2 \chi \\ \langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \rangle &= R^2 sin^2 \chi sin^2 \theta \end{aligned}$$

So, $ds^2 \simeq d\mathbf{S}^2$

$$= \begin{pmatrix} dR & d\chi & d\theta & d\phi \end{pmatrix} \begin{pmatrix} \langle \frac{\partial}{\partial R}, \frac{\partial}{\partial R} \rangle \mathbf{11} & 0 & 0 & 0 \\ 0 & \langle \frac{\partial}{\partial \chi}, \frac{\partial}{\partial \chi} \rangle \mathbf{e}_\chi \mathbf{e}_\chi & 0 & 0 \\ 0 & 0 & \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle \mathbf{e}_\theta \mathbf{e}_\theta & 0 \\ 0 & 0 & 0 & \langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \rangle \mathbf{e}_\phi \mathbf{e}_\phi \end{pmatrix} \begin{pmatrix} dR \\ d\chi \\ d\theta \\ d\phi \end{pmatrix}$$
$$= \begin{pmatrix} dR & d\chi & d\theta & d\phi \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & -R^2\mathbf{1} & 0 & 0 \\ 0 & 0 & -R^2 sin^2 \chi \mathbf{1} & 0 \\ 0 & 0 & 0 & -R^2 sin^2 \chi sin^2 \theta \mathbf{1} \end{pmatrix} \begin{pmatrix} dR \\ d\chi \\ d\theta \\ d\phi \end{pmatrix}$$

We can also describe $\mathbf{R}^4$ in (pseudo)-polar coordinates and get a metric analogous to the Robertson-Walker metric for hyperbolic coordinates:

Let

$$\begin{aligned} x &= Rcosh\chi \\ y &= Rsinh\chi cos\theta \\ z &= Rsinh\chi sin\theta cos\phi \\ \eta &= Rsinh\chi sin\theta sin\phi \end{aligned}$$

Using the chain rule:

$$\frac{\partial}{\partial R} = \frac{\partial x}{\partial R}\frac{\partial}{\partial x} + \frac{\partial y}{\partial R}\frac{\partial}{\partial y} + \frac{\partial z}{\partial R}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial R}\frac{\partial}{\partial \eta}$$

$$
\begin{aligned}
\frac{\partial}{\partial \chi} &= \frac{\partial x}{\partial \chi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \chi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \chi}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial \chi}\frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial \theta}\frac{\partial}{\partial \eta} \\
\frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial \phi}\frac{\partial}{\partial \eta}
\end{aligned}
$$

Now, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \eta}$ are orthogonal unit vectors in $\mathbf{R}^4$ so

$$
\begin{aligned}
\langle \frac{\partial}{\partial R}, \frac{\partial}{\partial R} \rangle &= 1 \\
\langle \frac{\partial}{\partial \chi}, \frac{\partial}{\partial \chi} \rangle &= R^2 \\
\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle &= R^2 sinh^2 \chi \\
\langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \rangle &= R^2 sinh^2 \chi sin^2 \theta
\end{aligned}
$$

So, $ds^2 \simeq d\mathbf{S}^2$

$$
= \begin{pmatrix} dR & d\chi & d\theta & d\phi \end{pmatrix} \begin{pmatrix} \langle \frac{\partial}{\partial R}, \frac{\partial}{\partial R} \rangle \mathbf{11} & 0 & 0 & 0 \\ 0 & \langle \frac{\partial}{\partial \chi}, \frac{\partial}{\partial \chi} \rangle \mathbf{e}_\chi \mathbf{e}_\chi & 0 & 0 \\ 0 & 0 & \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle \mathbf{e}_\theta \mathbf{e}_\theta & 0 \\ 0 & 0 & 0 & \langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \rangle \mathbf{e}_\phi \mathbf{e}_\phi \end{pmatrix} \begin{pmatrix} dR \\ d\chi \\ d\theta \\ d\phi \end{pmatrix}
$$

$$
= \begin{pmatrix} dR & d\chi & d\theta & d\phi \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & -R^2 \mathbf{1} & 0 & 0 \\ 0 & 0 & -R^2 sinh^2 \chi \mathbf{1} & 0 \\ 0 & 0 & 0 & -R^2 sinh^2 \chi sin^2 \theta \mathbf{1} \end{pmatrix} \begin{pmatrix} dR \\ d\chi \\ d\theta \\ d\phi \end{pmatrix}
$$

The above two are not metrics for Robertson-Walker space-time unless $Rcos\chi$ in the first and $Rcosh\chi$ in the second were proxies for time which they are not.

A theorem in Differential Topology states that an m dimensional manifold can be parameterized by up to $2m$ variables. In Schwarzschild coordinates t, R, θ, ϕ (which describe a spherically symmetric gravitational field):

$$\begin{aligned}
x &= x(t,R,\theta,\phi)\\
y &= y(t,R,\theta,\phi)\\
z &= z(t,R,\theta,\phi)\\
\eta &= \eta(t,R,\theta,\phi)\\
... &= ...\\
\xi &= \xi(t,R,\theta,\phi)
\end{aligned}$$

Using the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} + ... + \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}\\
\frac{\partial}{\partial R} &= \frac{\partial x}{\partial R}\frac{\partial}{\partial x} + \frac{\partial y}{\partial R}\frac{\partial}{\partial y} + \frac{\partial z}{\partial R}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial R}\frac{\partial}{\partial \eta} + ... + \frac{\partial \xi}{\partial R}\frac{\partial}{\partial \xi}\\
\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial \theta}\frac{\partial}{\partial \eta} + ... + \frac{\partial \xi}{\partial \theta}\frac{\partial}{\partial \xi}\\
\frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} + \frac{\partial \eta}{\partial \phi}\frac{\partial}{\partial \eta} + ... + \frac{\partial \xi}{\partial \phi}\frac{\partial}{\partial \xi}
\end{aligned}$$

Now, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \eta}, ..., \frac{\partial}{\partial \xi}$ are orthogonal unit vectors in $\mathbf{R}^n$ $(5 \leq n \leq 8)$ so

$ds^2 \simeq d\mathbf{S}^2$

$$= \begin{pmatrix} dt & dR & d\theta & d\phi \end{pmatrix} \begin{pmatrix} \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\rangle \mathbf{11} & 0 & 0 & 0 \\ 0 & \langle \frac{\partial}{\partial R}, \frac{\partial}{\partial R}\rangle \mathbf{e}_R\mathbf{e}_R & 0 & 0 \\ 0 & 0 & \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\rangle \mathbf{e}_\theta\mathbf{e}_\theta & 0 \\ 0 & 0 & 0 & \langle \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}\rangle \mathbf{e}_\phi\mathbf{e}_\phi \end{pmatrix} \begin{pmatrix} dt \\ dR \\ d\theta \\ d\phi \end{pmatrix}$$

$$= \begin{pmatrix} dt & dR & d\theta & d\phi \end{pmatrix} \begin{pmatrix} (1-\frac{\alpha}{R})\mathbf{1} & 0 & 0 & 0 \\ 0 & -(1-\frac{\alpha}{R})^{-1}\mathbf{1} & 0 & 0 \\ 0 & 0 & -R^2\mathbf{1} & 0 \\ 0 & 0 & 0 & -R^2 sin^2\theta\mathbf{1} \end{pmatrix} \begin{pmatrix} dt \\ dR \\ d\theta \\ d\phi \end{pmatrix}$$

where $\alpha = \frac{2GM}{c^2}$, $R = (r^3 + \alpha^3)^{1/3}$, and r is the Euclidean radius $r = \sqrt{x^2 + y^2 + z^2}$. In this case, $\langle \cdot, \cdot \rangle$ is the Euclidean metric in $\mathbf{R}^n$ and the latter equality was proved by Schwarzschild. The space $\mathbf{R}^n$ may or may not have physical 'reality' but here is simply a parameter space.

The equation from SR

$E^2/c^2 - p_x^2 - p_y^2 - p_z^2 = m^2c^2$ can be written as

$$m^2c^2\mathbf{1} = \begin{pmatrix} E/c & p_x & p_y & p_z \end{pmatrix} \begin{pmatrix} \mathbf{11} & 0 & 0 & 0 \\ 0 & \mathbf{ii} & 0 & 0 \\ 0 & 0 & \mathbf{jj} & 0 \\ 0 & 0 & 0 & \mathbf{kk} \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

and generalized to

$$m^2c^2\mathbf{1}$$

$$= \begin{pmatrix} E/c & p_R & p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} (1-\frac{\alpha}{R})\mathbf{1} & 0 & 0 & 0 \\ 0 & -(1-\frac{\alpha}{R})^{-1}\mathbf{1} & 0 & 0 \\ 0 & 0 & -R^2\mathbf{1} & 0 \\ 0 & 0 & 0 & -R^2 sin^2\theta\mathbf{1} \end{pmatrix} \begin{pmatrix} E/c \\ p_R \\ p_\theta \\ p_\phi \end{pmatrix}$$

Then

$$m^2c^4 = (1-\tfrac{\alpha}{R})E^2 - (1-\tfrac{\alpha}{R})^{-1}c^2p_R^2 - R^2c^2p_\theta^2 - R^2 sin^2\theta c^2 p_\phi^2$$

Though a 4-vector (A^0, A^1, A^2, A^3) is frame dependent, the 4-vector magnitude $||A||$ is preserved. It is for this reason the speed of light is an invariant. The traditional way of expressing this idea is

$$||A||^2 = g_{\alpha\beta}A^\alpha A^\beta = \overline{g}_{\gamma\delta}\overline{A}^\gamma\overline{A}^\delta.$$

For an object hovering in a gravitational field at radius R,

$E = \pm\frac{mc^2}{\sqrt{1-\frac{\alpha}{R}}}$. We can regard this as the energy in a bound system so we use the negative. Then $E = -\frac{mc^2}{\sqrt{1-\frac{\alpha}{R}}}$.

The energy that must be applied to remove the hovering mass to 'infinity' along a radial line must be

$E_\infty = mc^2 - \frac{mc^2}{\sqrt{1-\frac{\alpha}{R}}} = mc^2(1-\frac{1}{\sqrt{1-\frac{\alpha}{R}}})$ which is the gravitational binding energy in General Relativity.

This value is asymptotically equivalent for large R to the traditional Newtonian potential $-\frac{GMm}{r}$ and they are approximately equal for $R > 50\alpha$.

For an object at constant R and angular velocity p_ϕ/mR in a gravitational field

$$m^2c^2 = (1 - \frac{\alpha}{R})E^2/c^2 - p_\phi^2 R^2 sin^2\theta$$

$$E^2/c^2 = (1 - \frac{\alpha}{R})^{-1}(m^2c^2 + p_\phi^2 R^2 sin^2\theta)$$

$$E = \sqrt{(1 - \frac{\alpha}{R})^{-1}(m^2c^4 + p_\phi^2 c^2 R^2 sin^2\theta)}$$

For an object falling along a radial line with speed p_R/m

$$m^2c^2 = (1 - \frac{\alpha}{R})E^2/c^2 - (1 - \frac{\alpha}{R})^{-1}p_R^2$$

$$m^2c^2 + (1 - \frac{\alpha}{R})^{-1}p_R^2 = (1 - \frac{\alpha}{R})E^2/c^2$$

$$(1 - \frac{\alpha}{R})E^2 = m^2c^4 + (1 - \frac{\alpha}{R})^{-1}c^2p_R^2$$

$$E = \sqrt{(1 - \frac{\alpha}{R})^{-1}m^2c^4 + (1 - \frac{\alpha}{R})^{-2}c^2p_R^2}$$

and combining the two above cases

$$E = \sqrt{(1 - \frac{\alpha}{R})^{-1}(m^2c^4 + p_\phi^2 c^2 R^2 sin^2\theta) + (1 - \frac{\alpha}{R})^{-2}c^2p_R^2}$$

The geodesic equations

$\frac{d}{d\tau}[g_{ii}\frac{dx^i}{ds}] = \frac{1}{2}\Sigma_{j=0}^3 \partial_i g_{jj}\frac{dx^j}{ds}\frac{dx^j}{ds}$ for $0 \le i \le 3$ give the geodesic equations of motion with respect to the arclength parameter s.

$\partial_0 g_{jj} = 0$ for all j so $g_{00}\frac{dx^0}{ds} = constant = H$ where H is the total energy and can be identified as the Hamiltonian of the system.

$\partial_3 g_{jj} = 0$ for all j so $g_{33}\frac{dx^3}{ds} = constant = L$ where L is the angular momentum.

The gravitational field is spherically symmetric so we can transform the angle coordinates to our convenience. So, letting $\theta = \pi/2$ we have

$g_{33}\frac{dx^3}{ds} = R^2\frac{d\phi}{ds} = L$

For convenience in what follows let $\kappa_R = 1 - \frac{\alpha}{R}$

For a unit mass following a geodesic path

$\frac{d\gamma}{d\tau} = \frac{dt}{d\tau}\partial_t + \frac{dR}{d\tau}\partial_R + \frac{d\theta}{d\tau}\partial_\theta + \frac{d\phi}{d\tau}\partial_\phi$

The inner product $1 = \langle \frac{d\gamma}{d\tau}, \frac{d\gamma}{d\tau} \rangle$

$= (\frac{dt}{d\tau})^2\langle \partial_t, \partial_t \rangle + (\frac{dR}{d\tau})^2\langle \partial_R, \partial_R \rangle + (\frac{d\theta}{d\tau})^2\langle \partial_\theta, \partial_\theta \rangle + (\frac{d\phi}{d\tau})^2\langle \partial_\phi, \partial_\phi \rangle$

$= (H/\kappa_R)^2\kappa_R - (\frac{dR}{d\tau})^2\kappa_R^{-1} - (\frac{d\phi}{d\tau})^2R^2$

$= H^2\kappa_R^{-1} - (\frac{dR}{d\tau})^2\kappa_R^{-1} - (\frac{L}{R^2})^2R^2 = H^2\kappa_R^{-1} - (\frac{dR}{d\tau})^2\kappa_R^{-1} - \frac{L^2}{R^2}$

Then

$H^2\kappa_R^{-1} = 1 + (\frac{dR}{d\tau})^2\kappa_R^{-1} - \frac{L^2}{R^2}$

$H^2 = \kappa_R + (\frac{dR}{d\tau})^2 + \kappa_R\frac{L^2}{R^2}$

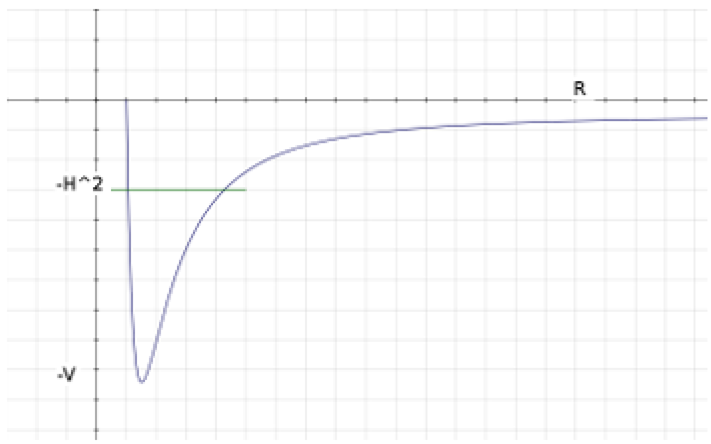
So, the *energy equation* is:

$H^2 = (\frac{dR}{d\tau})^2 + \kappa_R(1 + \frac{L^2}{R^2})$

We set $V(R) = \kappa_R(1 + \frac{L^2}{R^2}) = 1 - \frac{\alpha}{R} + \frac{L^2}{R^2} - \frac{\alpha L^2}{R^3}$

It depends only on R since L is a constant of the motion and $V(R)$ acts as the effective potential for H^2.

The graph below shows a generic plot for $-V$ and $-H^2$ as functions of R.



$-V \to -1$ as $R \to \infty$ so there are three cases:

(1) The body orbits the central gravitating body and its radius R varies between a minimum and maximum. $-H^2 < -1$

(2) The body has exactly escape velocity and $-H^2 = -1$

(3) The body has greater than escape velocity and is unbounded with $-H^2 > -1$

A body of unit mass starting at rest arbitrarily far from the center of symmetry and free falling along a radial line begins with $H^2 = 1$ (using units where $c = 1$).

At R its speed is given by $1 = (\frac{dR}{d\tau})^2 + 1 - \frac{\alpha}{R}$. As $R \to \alpha$, $(\frac{dR}{d\tau})^2 \to 1$. That is, the speed approaches c. We can also compute escape velocity at R. $(\frac{dR}{d\tau})^2 = \frac{\alpha}{R} = \frac{2GM}{R}$ and it follows that $\frac{1}{2}(\frac{dR}{d\tau})^2 = \frac{GM}{R}$ which is nearly the Newtonian value $\frac{1}{2}(\frac{dr}{d\tau})^2 = \frac{GM}{r}$ except $R = (r^3 + \alpha^3)^{1/3}$.

Preservation of Quaternion Structure:

In general, we can say that relativity theory is essentially a theory about how Nature preserves Quaternion structure in its operations. For example, the energy-momentum 4-vector behaves as a contravariant vector where the preservation of its magnitude-squared $E^2/c^2 - p_1^2 - p_2^2 - p_3^2$ with respect to different coordinates systems represents the preservation of quaternion structure. However, **it should be noted that not all pairs of coordinate systems have a metric preserving transformation linking them**.

Consider the metric representation

$d\tau^2$

$= (1 - \frac{\alpha R}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\theta^2 - (R^2 + \alpha^2 + \frac{\alpha R \alpha^2}{\rho^2 sin^2\theta})sin^2\theta d\phi^2 + \frac{2\alpha R \alpha sin^2\theta}{\rho^2}dtd\phi$

with $\alpha = \frac{J}{Mc}$ where J is the angular momentum; $\rho^2 = R^2 + \alpha^2 cos^2\theta$ where R is the area radius; and $\Delta = R^2 - \alpha R + \alpha^2$

This is called the Kerr metric and describes space-time in the vicinity of a rotating axially symmetric gravitational body.

We can see that due to the presence of the term $dtd\phi$, this metric is not in orthogonal form. To find a parameterization we would first need to put the matrix

$$\begin{pmatrix} (1 - \frac{\alpha R}{\rho^2}) & 0 & 0 & \frac{\alpha R \alpha sin^2\theta}{\rho^2} \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{\alpha R \alpha sin^2\theta}{\rho^2} & 0 & 0 & -(R^2 + \alpha^2 + \frac{\alpha R \alpha^2}{\rho^2 sin^2\theta})sin^2\theta \end{pmatrix}$$

in diagonal form. However, for small M we can neglect the α term and get the metric representation

$d\tau^2 = dt^2 - \frac{\rho^2}{R^2+\alpha^2}dR^2 - \rho^2 d\theta^2 - (R^2 + \alpha^2)sin^2\theta d\phi^2$

with the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\rho^2}{R^2+\alpha^2} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & (R^2 + \alpha^2)sin^2\theta \end{pmatrix} = [\frac{\partial(t,x,y,z)}{\partial(t,R,\theta,\phi)}]^T \frac{\partial(t,x,y,z)}{\partial(t,R,\theta,\phi)}$$

using the parameterization

$$t = t$$

$$\begin{aligned} x &= \sqrt{R^2+\alpha^2} sin\theta cos\phi \\ y &= \sqrt{R^2+\alpha^2} sin\theta sin\phi \\ z &= R cos\theta \end{aligned}$$

That is,

$$\begin{pmatrix} ||\frac{\partial}{\partial t}||^2 & 0 & 0 & 0 \\ 0 & ||\frac{\partial}{\partial R}||^2 & 0 & 0 \\ 0 & 0 & ||\frac{\partial}{\partial \theta}||^2 & 0 \\ 0 & 0 & 0 & ||\frac{\partial}{\partial \phi}||^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\rho^2}{R^2+\alpha^2} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & (R^2+\alpha^2) sin^2\theta \end{pmatrix}$$

A metric representation in matrix form with real entries

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

must be symmetric since it defines a symmetric bilinear form. A standard result in linear algebra is that real symmetric matrices can be diagonalized. For a smoothly varying metric representation there must be a smoothly varying orthogonal Q such that

$$\begin{pmatrix} g'_{00} & 0 & 0 & 0 \\ 0 & g'_{11} & 0 & 0 \\ 0 & 0 & g'_{22} & 0 \\ 0 & 0 & 0 & g'_{33} \end{pmatrix} = Q^T \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} Q,$$

If g' has signature (+,-,-,-) then there is a quaternion basis $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the coordinates of g' so that

$$\begin{pmatrix} g'_{00}\mathbf{1} & 0 & 0 & 0 \\ 0 & g'_{11}\mathbf{1} & 0 & 0 \\ 0 & 0 & g'_{22}\mathbf{1} & 0 \\ 0 & 0 & 0 & g'_{33}\mathbf{1} \end{pmatrix}$$

$$= \begin{pmatrix} ||\frac{\partial}{\partial x_0}||\mathbf{1} & 0 & 0 & 0 \\ 0 & ||\frac{\partial}{\partial x_1}||\mathbf{e}_1 & 0 & 0 \\ 0 & 0 & ||\frac{\partial}{\partial x_2}||\mathbf{e}_2 & 0 \\ 0 & 0 & 0 & ||\frac{\partial}{\partial x_3}||\mathbf{e}_3 \end{pmatrix}^T$$

$$\times \begin{pmatrix} \|\frac{\partial}{\partial x_0}\|\mathbf{1} & 0 & 0 & 0 \\ 0 & \|\frac{\partial}{\partial x_1}\|\mathbf{e}_1 & 0 & 0 \\ 0 & 0 & \|\frac{\partial}{\partial x_2}\|\mathbf{e}_2 & 0 \\ 0 & 0 & 0 & \|\frac{\partial}{\partial x_3}\|\mathbf{e}_3 \end{pmatrix}$$

where the $\frac{\partial}{\partial x_\mu}$; $\mu = 0, 1, 2, 3$ are the coordinate tangent vectors.

The symmetric bilinear form for the Kerr metric

$$K = \begin{pmatrix} (1 - \frac{\alpha R}{\rho^2}) & 0 & 0 & \frac{\alpha R \alpha sin^2\theta}{\rho^2} \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{\alpha R \alpha sin^2\theta}{\rho^2} & 0 & 0 & -(R^2 + \alpha^2 + \frac{\alpha R \alpha^2}{\rho^2 sin^2\theta})sin^2\theta \end{pmatrix}$$

we can write as $K = \begin{pmatrix} A & 0 & 0 & E \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ E & 0 & 0 & D \end{pmatrix}$

The eigenvalues for K are $\lambda = \frac{(A+D)\pm\sqrt{(A-D)^2+4E^2}}{2}, B, C$.

Then the diagonal bilinear form for the Kerr metric is

$$Q^T K Q = \begin{pmatrix} \frac{(A+D)+\sqrt{(A-D)^2+4E^2}}{2} & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & \frac{(A+D)-\sqrt{(A-D)^2+4E^2}}{2} \end{pmatrix}$$

$B < 0$ and $C < 0$ so quaternion structure is preserved if

$$\frac{(A + D) + \sqrt{(A - D)^2 + 4E^2}}{2} > 0$$

and

$$\frac{(A + D) - \sqrt{(A - D)^2 + 4E^2}}{2} < 0$$

.

The Minkowski Metric (continued):

We denote $\sqrt{\mathbf{g}} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ 0 & 0 & \mathbf{j} & 0 \\ 0 & 0 & 0 & \mathbf{k} \end{pmatrix}$

Then the Minkowski metric can be written

$g(\mathbf{V'}, \mathbf{V'}) = (\sqrt{\mathbf{g}}\mathbf{V'})^T \sqrt{\mathbf{g}}\mathbf{V'}$ where $\mathbf{V'}$ is a vector in the tangent space expressed in Minkowski coordinates.

Let (t', u, v, w) be alternate coordinates, not necessarily isometric to Minkowski coordinates.

We denote $\sqrt{\mathbf{g'}} = \begin{pmatrix} ||\frac{\partial}{\partial t'}||\mathbf{1} & 0 & 0 & 0 \\ 0 & ||\frac{\partial}{\partial u}||\mathbf{e}_u & 0 & 0 \\ 0 & 0 & ||\frac{\partial}{\partial v}||\mathbf{e}_v & 0 \\ 0 & 0 & 0 & ||\frac{\partial}{\partial w}||\mathbf{e}_w \end{pmatrix}$

Then $g'(\mathbf{V}, \mathbf{V}) = (\sqrt{\mathbf{g'}}\mathbf{V})^T \sqrt{\mathbf{g'}}\mathbf{V}$ where $\mathbf{V}$ is the 4-vector $\mathbf{V'}$ in the alternate coordinates. The invariance of the 4-vector magnitude requires that $g(\mathbf{V'}, \mathbf{V'}) = g'(\mathbf{V}, \mathbf{V})$. That is,

$$||\mathbf{V}||^2 = \begin{pmatrix} V'^0 & V'^1 & V'^2 & V'^3 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & -\mathbf{1} & 0 & 0 \\ 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} V'^0 \\ V'^1 \\ V'^2 \\ V'^3 \end{pmatrix}$$

$$= \begin{pmatrix} V^0 & V^1 & V^2 & V^3 \end{pmatrix} \begin{pmatrix} ||\frac{\partial}{\partial t'}||^2\mathbf{1} & 0 & 0 & 0 \\ 0 & -||\frac{\partial}{\partial u}||^2\mathbf{1} & 0 & 0 \\ 0 & 0 & -||\frac{\partial}{\partial v}||^2\mathbf{1} & 0 \\ 0 & 0 & 0 & -||\frac{\partial}{\partial w}||^2\mathbf{1} \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

We can also write the above equation(s) as

$$||\mathbf{V}||^2 = (V'^0\mathbf{1})^2 + (V'^1\mathbf{i} + V'^2\mathbf{j} + V'^3\mathbf{k})^2$$

$$= (V^0||\frac{\partial}{\partial t'}||\mathbf{1})^2 + (V^1||\frac{\partial}{\partial u}||\mathbf{e}_u + V^2||\frac{\partial}{\partial v}||\mathbf{e}_v + V^3||\frac{\partial}{\partial w}||\mathbf{e}_w)^2$$

Footnote*:

Using the transformation

$$\begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

we can derive formulas for velocity addition and the relativistic Doppler effect:

Let S, S', S'' be frames moving with uniform velocity along the x-direction. Let S' be moving with velocity v with respect to S and S'' be moving with velocity w with respect to S'. First note that $\frac{sinh(\alpha)}{cosh(\alpha)} = v$ and $\frac{sinh(\beta)}{cosh(\beta)} = w$. Then

$$\begin{pmatrix} \Delta t'' \\ \Delta x'' \\ \Delta y'' \\ \Delta z'' \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cosh(\beta) & sinh(\beta) & 0 & 0 \\ sinh(\beta) & cosh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

.

$$= \begin{pmatrix} cosh(\alpha)cosh(\beta) + sinh(\alpha)sinh(\beta) & cosh(\alpha)sinh(\beta) + sinh(\alpha)cosh(\beta) & 0 & 0 \\ cosh(\alpha)sinh(\beta) + sinh(\alpha)cosh(\beta) & cosh(\alpha)cosh(\beta) + sinh(\alpha)sinh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

$$= \begin{pmatrix} cosh(\alpha+\beta) & sinh(\alpha+\beta) & 0 & 0 \\ sinh(\alpha+\beta) & cosh(\alpha+\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

The combined velocity is

$\frac{sinh(\alpha+\beta)}{cosh(\alpha+\beta)} = \frac{cosh(\alpha)sinh(\beta)+sinh(\alpha)cosh(\beta)}{cosh(\alpha)cosh(\beta)+sinh(\alpha)sinh(\beta)}$ and dividing top and bottom by $cosh(\alpha)cosh(\beta)$ gives the combined velocity $\frac{sinh(\alpha+\beta)}{cosh(\alpha+\beta)} = \frac{v+w}{1+vw}$.

Now consider a pulse of light with wavelength measured at S to be λ_e and travelling in the direction of increasing x. Measured w.r.t. S, one cycle completes in Δt. The distance of the next wave front from O' is

$\lambda_e + v\Delta t = \Delta t$. Then $\Delta t = \frac{\lambda_e}{1-v}$. Then

$$\begin{pmatrix} \Delta t \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t' \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $\lambda_o = \Delta t' = \frac{\Delta t}{cosh(\alpha)} = \frac{\lambda_e}{(1-v)cosh(\alpha)} = \lambda_e \sqrt{\frac{1+v}{1-v}} = \lambda_e \sqrt{\frac{cosh(\alpha)+sinh(\alpha)}{cosh(\alpha)-sinh(\alpha)}}$ where λ_o is the observed wavelength.

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