The Schwarzschild Metric

Our current formulation of the spherically symmetric solution around a gravitating body is:

$$ds^2 = (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2 - r^2 d\Omega^2$$

where c is set at 1, t is the elapsed time of a clock 'at infinity', r is the scalar distance, r_s is the 'event horizon' radius $2GM$, and $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$

In Schwarzschild's original paper from 1916 he does not use r the same way. His equation is:

$$ds^2 = (1 - \alpha/R)dt^2 - (1 - \alpha/R)^{-1}dR^2 - R^2 d\Omega^2$$

where $\alpha = 2GM$, $R = (r^3 + \alpha^3)^{1/3}$ and $r = \sqrt{x^2 + y^2 + z^2}$ is zero at the center of symmetry (the origin). $R = \sqrt{\frac{A}{4\pi}}$ is the area radius of a sphere centered at $r = 0$. If r is the Euclidean distance we could have $R < r$ (positive spatial curvature), $R = r$ (zero spatial curvature), or $R > r$ (negative spatial curvature). In our case here we have $R > r$ which implies negative spatial curvature. One implication is that the event horizon at α gets removed which affects the theory of black holes.

If we set

$$d\sigma^2 = R^2 d\Omega^2 = R^2(d\theta^2 + sin^2\theta d\phi^2)$$

we get a metric on the 2-sphere of radius $R = (r^3 + \alpha^3)^{1/3}$ where r as before is the scalar radius. From this we can compute that the circumference $C = 2\pi R = 2\pi(r^3 + \alpha^3)^{1/3} > 2\pi r$ and the surface area $A = 4\pi R^2 = 4\pi(r^3 + \alpha^3)^{2/3} > 4\pi r^2$. This can only happen if space itself is negatively curved. Not only is space-time curved but space *itself* is curved negatively. That is, $(r^3 + \alpha^3)^{2/3}/r^2 \to \infty$ as $r \to 0$.

Considering space as a foliation of 3D "hyperboloids" (indexed by α) given by

$$u = (r^3 + \alpha^3)^{1/3} sin\theta cos\phi$$

$$\begin{aligned} v &= (r^3+\alpha^3)^{1/3} sin\theta sin\phi \\ w &= r \end{aligned}$$

where $r > 0$, $0 < \theta < \pi$ and $0 \leq \phi < 2\pi$. The angle θ is the amount of deflection from the polar axis and ϕ is the longitudinal angle. Each leaf in the foliation has a metric

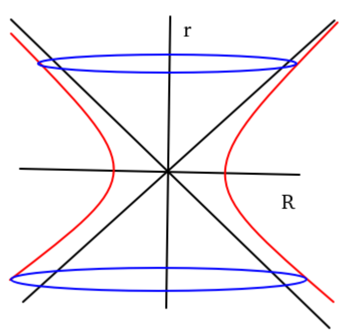
$$ds^2 = ((R^3-\alpha^3)^{-4/3}R^4 + sin^2\theta)dR^2 + R^2(cos^2\theta d\theta^2 + sin^2\theta d\phi^2)$$

with $R = (r^3+\alpha^3)^{1/3}$.*

If we set $\theta = \frac{\pi}{2}$ then $\{((r^3+\alpha^3)^{1/3}, \frac{\pi}{2}, \phi) : r > 0 \ and \ 0 \leq \phi < 2\pi\}$ is a 2-surface of revolution with

$$\begin{aligned} u &= (r^3+\alpha^3)^{1/3} cos\phi \\ v &= (r^3+\alpha^3)^{1/3} sin\phi \\ w &= r \end{aligned}$$

as shown in the following figure



We can easily compute its Gaussian curvature K at $r = \alpha$:

For the 1st fundamental form:

$$E = (r^3+\alpha^3)^{2/3}$$

$$
\begin{aligned}
F &= 0 \\
G &= \frac{r^4}{(r^3+\alpha^3)^{4/3}} + 1
\end{aligned}
$$

and the 2nd fundamental form:

$$
\begin{aligned}
e &= -\frac{(r^3+\alpha^3)^{1/3}}{\sqrt{\frac{r^4}{(r^3+\alpha^3)^{4/3}}+1}} \\
f &= 0 \\
g &= \frac{2r(r^3+\alpha^3)^{2/3} - 3r^4(r^3+\alpha^3)^{-1/3}}{(r^3+\alpha^3)^{4/3}} \Big/ \sqrt{\frac{r^4}{(r^3+\alpha^3)^{4/3}}+1}
\end{aligned}
$$

$\kappa_1 = g/G$ and $\kappa_2 = e/E$

At $r = \alpha$, $\kappa_1 = g/G = \frac{2r(r^3+\alpha^3)^{2/3} - 2r^4(r^3+\alpha^3)^{-1/3}}{(r^3+\alpha^3)^{4/3}} / \left(\frac{r^4}{(r^3+\alpha^3)^{4/3}} + 1\right)^{3/2}$
$= \frac{2\alpha(2\alpha^3)^{2/3} - 2\alpha^4(2\alpha^3)^{-1/3}}{(2\alpha^3)^{4/3}} / \left(\frac{\alpha^4}{(2\alpha^3)^{4/3}} + 1\right)^{3/2}$

$= \frac{2(2)^{2/3} - 2(2)^{-1/3}}{\alpha(2)^{4/3}} / \left(\frac{1}{(2)^{4/3}} + 1\right)^{3/2} > 0$

$= \frac{4-2}{\alpha(2)^{5/3}} / \left(\frac{1}{(2)^{4/3}} + 1\right)^{3/2} > 0$

At $r = \alpha$, $\kappa_2 = e/E = -\frac{(r^3+\alpha^3)^{1/3}}{\sqrt{\frac{r^4}{(r^3+\alpha^3)^{4/3}}+1}} / (r^3+\alpha^3)^{2/3}$
$= -\frac{(2)^{1/3}}{\sqrt{\frac{1}{(2)^{4/3}}+1}} / 2^{2/3}\alpha < 0$

So, at $r = \alpha$, $K = \kappa_1\kappa_2 < 0$ which shows the surface is negatively curved**.

The uniqueness of the Schwarzschild (1916) solution arises directly out of the equations themselves and does not depend on an external theorem to guarantee uniqueness. His derivation implies that space is negatively curved.

The term α arises as a constant of integration in the 1916 derivation so we must compute it empirically.

The author in [Sch 1916] derives $\dot{\phi}^2 = \alpha/2(r^3 + \alpha^3) = \alpha/2R^3$

The equation $2\dot{\phi}^2 R^3 = \alpha$ imposes a constraint on $\dot{\phi}$ given R (and implicitly r) similar to Kepler's 3rd law: $T^2 = Kr^3$. For a circular orbit, $K = 4\pi^2/GM$

We can rewrite Kepler's 3rd law as $(2\pi/\dot{\phi})^2 = Kr^3$ and then $8\pi^2/K = 2\dot{\phi}^2 r^3$

So, we have

$$\begin{aligned} 2\dot{\phi}^2 r^3 &= 8\pi^2/K \qquad (1) \\ 2\dot{\phi}^2 R^3 &= \alpha \qquad (2) \end{aligned}$$

with equation (1) being Kepler's 3rd law and equation (2) being the relativistic counterpart.

Since α is a constant we have $R/r \to 1$ as $r \to \infty$. Then $\alpha K/8\pi^2 = 1$ and so $\alpha = 8\pi^2/K = 2GM$ presumably in units where $c = 1$.

Let $A = (A^0, A^1, A^2, A^3)$ be a 4-vector with pseudo-norm

$$||A|| = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$$

which can be expressed as

$$||A|| = \begin{pmatrix} A^0 & A^1 & A^2 & A^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

Suppose A is Lorentz invariant. Then

$$||A|| = \begin{pmatrix} A^0 & A^1 & A^2 & A^3 \end{pmatrix} \Lambda^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Lambda \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

Let A be expressed in spherical coordinates $A` = (A^{`0}, A^{`1}, A^{`2}, A^{`3})$ with the transformation

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = M \begin{pmatrix} A^{`0} \\ A^{`1} \\ A^{`2} \\ A^{`3} \end{pmatrix}$$

Then

$$||A|| = \begin{pmatrix} A^{`0} & A^{`1} & A^{`2} & A^{`3} \end{pmatrix} M^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} M \begin{pmatrix} A^{`0} \\ A^{`1} \\ A^{`2} \\ A^{`3} \end{pmatrix}$$

and

$$||A|| = \begin{pmatrix} A^{`0} & A^{`1} & A^{`2} & A^{`3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix} \begin{pmatrix} A^{`0} \\ A^{`1} \\ A^{`2} \\ A^{`3} \end{pmatrix}$$

A 4-vector of interest is $(E^2/c^2, P_x^2, P_y^2, P_z^2)$ with pseudo-norm m^2c^2

It can be written as $m^2c^2 = \begin{pmatrix} E/c & P_x & P_y & P_z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} E/c \\ P_x \\ P_y \\ P_z \end{pmatrix}$

and expressing this in spherical coordinates

$$m^2c^2 = \begin{pmatrix} E/c & P_r & P_\theta & P_\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix} \begin{pmatrix} E/c \\ P_r \\ P_\theta \\ P_\phi \end{pmatrix}$$

Gravity induces a perturbation of the metric giving us

$$m^2c^2 = \begin{pmatrix} E/c & P_r & P_\theta & P_\phi \end{pmatrix} \begin{pmatrix} (1-\frac{\alpha}{R}) & 0 & 0 & 0 \\ 0 & -(1-\frac{\alpha}{R})^{-1} & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2 sin^2\theta \end{pmatrix} \begin{pmatrix} E/c \\ P_r \\ P_\theta \\ P_\phi \end{pmatrix}$$

and given that ∂_r and ∂_R are colinear (hence $P_r = P_R$), therefore

$m^2c^2 = (1-\frac{\alpha}{R})E^2/c^2 - (1-\frac{\alpha}{R})^{-1}P_R^2 - R^2P_\theta^2 - R^2 sin^2\theta P_\phi^2$

For an object hovering in a gravitational field at radius R

$m^2c^2 = (1-\frac{\alpha}{R})E^2/c^2$

$E^2/c^2 = (1-\frac{\alpha}{R})^{-1}m^2c^2$

For an object at constant R and angular velocity $(P_\phi/m)R^2$ in a gravitational field

$m^2c^2 = (1-\frac{\alpha}{R})E^2/c^2 - P_\phi^2R^2$

$E^2/c^2 = (1-\frac{\alpha}{R})^{-1}(m^2c^2 + P_\phi^2R^2)$

For an object falling along a radial line with speed P_R/m

$m^2c^2 = (1-\frac{\alpha}{R})E^2/c^2 - (1-\frac{\alpha}{R})^{-1}P_R^2$

$m^2c^2 + (1-\frac{\alpha}{R})^{-1}P_R^2 = (1-\frac{\alpha}{R})E^2/c^2$

$(1-\frac{\alpha}{R})E^2 = m^2c^4 + (1-\frac{\alpha}{R})^{-1}c^2P_R^2$

$E^2/c^2 = (1-\frac{\alpha}{R})^{-1}(m^2c^2 + (1-\frac{\alpha}{R})^{-1}P_R^2)$

and combining the two above cases

$E^2/c^2 = (1-\frac{\alpha}{R})^{-1}(m^2c^2 + (1-\frac{\alpha}{R})^{-1}P_R^2 + P_\phi^2R^2)$

The energy that must be applied to remove the hovering mass to 'infinity' must be the negative of

$E_\infty = mc^2 - \frac{mc^2}{\sqrt{1-\frac{\alpha}{R}}} = mc^2(1-\frac{1}{\sqrt{1-\frac{\alpha}{R}}})$ which is the gravitational binding energy in General Relativity.

This value is asymptotically equivalent for large R to the traditional Newtonian potential $-\frac{GMm}{R}$ and they are approximately equal for $R > 50\alpha$.

To calculate E in the above cases we take the negative square root for a bound test particle m and positive for the unbound. That is based on the convention that gravitational binding energy is negative.

We can easily see that the above formula for E_∞ refutes the cult-like dogma of Black Hole event horizons. For, a particle falling towards the gravitating center gains infinite energy which, having fallen through the event horizon, is transferred to the mass of the gravitating body. Even one atom falling through the event horizon gives the Black Hole infinite mass.

The Energy Equation:

It can be shown that $E = (1 - \frac{\alpha}{R})dt/ds$ and $L = R^2 d\phi/ds$ are constants of the motion and are designated energy and angular momentum respectively.

It can also be shown that *The Energy Equation**** satisfies

$E^2 = (dR/ds)^2 + (L^2/R^2)(1 - \frac{\alpha}{R})$.

The Speed of Light in a Gravitational Field:

Radial:

Setting $L = 0$ above we have $E^2 = (dR/ds)^2$.

Then $(1 - \frac{\alpha}{R})^2 dt^2/ds^2 = dR^2/ds^2$.

That is, $(1 - \frac{\alpha}{R}) = dR/dt$.

Angular:

Setting $dR/ds = 0$ in the energy equation gives

$(1 - \frac{\alpha}{R})^2 dt^2/ds^2 = R^2(d\phi^2/ds^2)(1 - \frac{\alpha}{R})$

Then $(1 - \frac{\alpha}{R}) = R^2 d\phi^2/dt^2$

Then $\sqrt{(1 - \frac{\alpha}{R})} = Rd\phi/dt$

In both cases, relative to a clock 'at infinity', the speed of light appears slower in the gravitational field.

Photo-spheres:

From above we have $R^2\dot{\phi}^2 = (1 - \frac{\alpha}{R})$ and from Sch[1916], $\dot{\phi}^2 = \frac{\alpha}{2R^3}$.

Then $\frac{\alpha}{2R} = (1 - \frac{\alpha}{R})$ and $R = \frac{3}{2}\alpha$.

At such a radius photons can be trapped in an orbit around a mass concentrated at (or near) the origin.

The Kruskal Extension:

We have seen above that the area radius $R \to \alpha$ as $r \to 0$

This implies that the ratio of the surface areas $\frac{4\pi R^2}{4\pi r^2} \to \infty$ as $r \to 0$

We saw above that this is easily accounted for by the negative curvature around the central mass (there assumed to be concentrated at $r = 0$).

There is another interpretation that leads to the cult-like dogma of Black Hole *event horizons*.

Suppose we let r take on negative values so that $R = 0$ when $r = -\alpha$. Then $r = \sqrt{x^2 + y^2 + z^2} < 0$ which implies imaginary values for the Cartesian coordinates x, y, z.

The Black Hole *event horizon* is at $r = 0$ where $R = \alpha$. In this view the central mass is concentrated at $R = 0$, that is at $r = -\alpha$, which, though a point at the center of symmetry, has extension in imaginary space = $span\{ix, iy, iz\}$

For $R < \alpha$, that is for $r < 0$, the three dimensions of space become time-like and the one dimension of time becomes space-like.

If we allow $R < 0$ then we encounter the idea of space-time wormholes, interesting for science fiction but not for actual science.

Kruskal coordinates are defined as follows:

Setting $\alpha = \alpha = 2GM$, the Schwarzschild coordinates t, R, θ, ϕ are transformed by

$$T = (\frac{R}{\alpha} - 1)^{1/2} e^{R/2\alpha} sinh(\frac{t}{2\alpha})$$

$$X = (\frac{R}{\alpha} - 1)^{1/2} e^{R/2\alpha} cosh(\frac{t}{2\alpha})$$

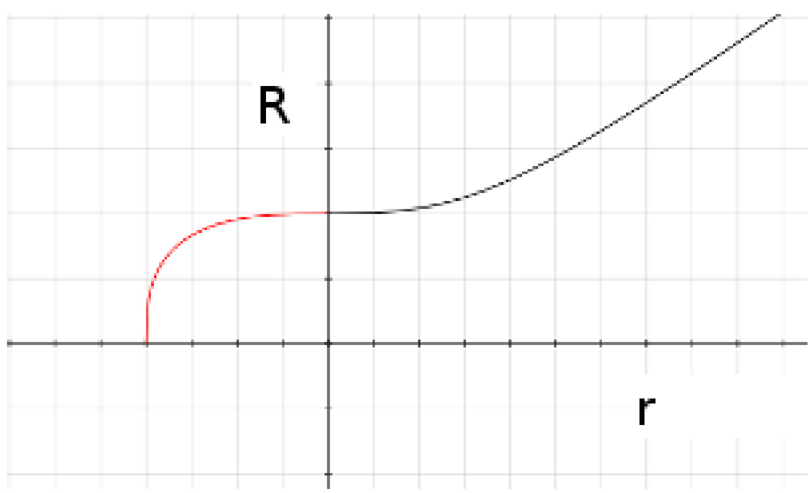
Then the Schwarzschild metric representation is transformed to

$$ds^2 = \frac{4\alpha^3}{R} e^{-R/\alpha}(dT^2 - dX^2) - R^2 d\Omega^2.$$

Those devoted to Black Hole event horizons speak of an inside where $0 < R < \alpha$ and an outside $\alpha < R$. However, there is no inside since $R = (r^3 + \alpha^3)^{1/3} > \alpha$ unless, as indicated above, r is allowed to become negative.

A trick that is used involves replacing R with r and pretending it is still a solution to the field equations of GR. Then one can speak about an inside $0 < r < \alpha$ and an outside $\alpha < r$ without requiring that r become negative.

The figure below shows the relation between r and R. The black graph shows the Schwarzschild solution (when vertical axis is at $r = 0$). The red graph shows the Kruskal extension (when vertical axis is at $r = \alpha$).



All observable phenomena for $R > \alpha$ only require for their explanation the Schwarzschild solution (the black part of the graph). The red part of the graph, even if mathematically valid, is not required to explain any physically observable phenomena and therefore counts as inventing unnecessary entities - a violation of Occam's Razor.

So, what exactly is a Black Hole? Imagine that a star, after depleting its nuclear fuel, can no longer resist the compression effect of its own gravitation and shrinks down to some limiting density at $r = \epsilon$. Its scalar radius is $r = \epsilon$ but negative curvature exaggerates its area radius $R_\epsilon = (\epsilon^3 + \alpha^3)^{1/3}$ where $\alpha = 2GM$. It would still emit black body radiation depending on its temperature but that would be red-shifted, perhaps extremely so, and dim making it appear as a black disk. This would be due to two processes. The first would be time dilation. The second would be reduction in electromagnetic flux because of the extreme negative curvature. If you were to observe it from outside any of its photo-spheres and against a background of stars, you would observe a black disk if its emitted radiation were sufficiently red-shifted and the flux sufficiently reduced. If you were to observe a spacecraft falling towards it emitting a signal at regular time intervals then to you, the outside observer, such intervals would get stretched out due to time dilation. The falling spacecraft, if it could not escape, would eventually crash into the mass at the center. It would not cross a fictional 'event horizon' before crashing into the center. Before crashing into the central mass the light from the compressed star would become more visible to the falling spacecraft due to the reduction in red-shift.

We propose as a definition for a Black Hole *a mass compressed into a space smaller than its photo-sphere*. That is, letting $\epsilon =$ the scalar radius of the central mass, $R_\epsilon = (\epsilon^3 + \alpha^3)^{1/3} < \frac{3}{2}\alpha$. Then $\epsilon < \frac{\sqrt[3]{19}}{2}\alpha \approx 1.334\alpha$.

The idea that there are values of $R = (r^3 + \alpha^3)^{1/3} < \alpha$ is a mistaken one because $r = \sqrt{x^2 + y^2 + z^2} \geq 0$ and $\alpha = 2GM > 0$. Such an ongoing conceptual error leads to the idea of a Black Hole event horizon that has become a major defect in modern astrophysics. Perhaps for some who should know better, the idea of a Black Hole event horizon has become too good a joke to spoil!

* Footnote: Foliation of "Hyperboloids":

As stated above, space around a non-rotating gravitational body can be viewed as a foliation of "hyperboloids"

$$\begin{aligned} u &= Rcos\phi = (r^3+\alpha^3)^{1/3}cos\phi \\ v &= Rsin\phi = (r^3+\alpha^3)^{1/3}sin\phi \\ w &= r \end{aligned}$$

where $\alpha \geq 0$, $r > 0$, and $0 \leq \phi < 2\pi$. The angle ϕ is the longitudinal angle. Each "hyperboloid" in the foliation is a surface of revolution of $(r^3+\alpha^3)^{1/3}$ about the r axis. Each leaf in the foliation (indexed by α) has a surface metric $dS^2 = ((R^3-\alpha^3)^{-4/3}R^4+1)dR^2+R^2d\phi^2$ with $R = (r^3+\alpha^3)^{1/3}$ and $\theta = \frac{\pi}{2}$. They are surfaces of revolution around the r-axis each determined by the parameter $\alpha = 2GM$. So, around a given gravitating body there is a unique α but the polar axis is arbitrary since the body does not rotate.

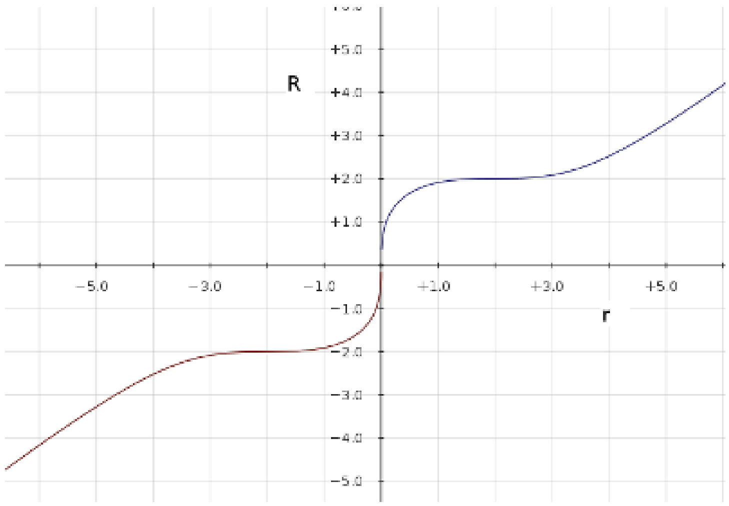
Wormholes: A Potential Gravitational Anomaly

In the foregoing, we have shown that there is no coordinate singularity corresponding to an event horizon around a gravitating body. A gravitating body has one singularity at the origin (the *nullpunkt*)as Schwarzschild vigorously emphasized (See Schwarzschild [1916]).

However, suppose there is no gravitating body causing the gravitational field. At first this seems bizarre if not impossible. Nevertheless, the mathematics allows for it. In the derivation of the metric which bears his name, Schwarzschild found a constant of integration ρ. The constant corresponding to the Schwarzschild radius he called α. He reasoned that in order for the singularity to be at the origin we must have $\rho = \alpha^3$. In other words, what we assign to ρ determines where the singularity will be.

As a disclaimer, we emphasize that in the following we are discussing an artifact of the mathematics without, necessarily, any physical reality.

Letting $\rho = 0$ we get a coordinate singularity at $r = \alpha$ with $R = ((r-\alpha)^3 + \alpha^3)^{1/3}$ for $r > 0$. With $\alpha = 2$ and $R = -(-(r-\alpha)^3 + \alpha^3)^{1/3}$ for $r < 0$ we get the graphs



A unit mass would fall towards a (non-existent) Schwarzschild mass of $M = \alpha/2G$ and encounter three singularities before emerging out the other side where $R < -\alpha$. Whether indeed such an entity could have physical existence we cannot verify without falling through one. From the outside it would appear as though an actual mass were present causing the gravity. A clock on a body falling into one would have no different behavior than a clock falling toward an actual gravitating mass. In both cases the clock would appear from the outside to slow down and approach a full stop as it got closer to $r = \alpha$ (wormhole) or $r = 0$ (gravitating body).

Mathematically speaking, a geodesic cannot cross a singularity. A coordinate transformation cannot fix this by removing the singularity without destroying the original metric which is a solution to the field equation(s).

** Footnote: Schwarzschild Spatial Curvature

Using the Schwarzschild metric:

$$g = \begin{pmatrix} 1-\alpha/R & 0 & 0 & 0 \\ 0 & -(1-\alpha/R)^{-1} & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2 sin^2\theta \end{pmatrix}$$

$$g = \begin{pmatrix} 1-\frac{\alpha}{(r^3+\alpha^3)^{1/3}} & 0 & 0 & 0 \\ 0 & -(1-\frac{\alpha}{(r^3+\alpha^3)^{1/3}})^{-1} & 0 & 0 \\ 0 & 0 & -(r^3+\alpha^3)^{2/3} & 0 \\ 0 & 0 & 0 & -(r^3+\alpha^3)^{2/3} sin^2\theta \end{pmatrix}$$

We are interested here only in the spatial curvature so we take g to be

$$g = \begin{pmatrix} (1-\frac{\alpha}{(r^3+\alpha^3)^{1/3}})^{-1} & 0 & 0 \\ 0 & (r^3+\alpha^3)^{2/3} & 0 \\ 0 & 0 & (r^3+\alpha^3)^{2/3} sin^2\theta \end{pmatrix}$$

$\Gamma^k_{ij} = \frac{1}{2} g^{km}(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m})$

To determine the spatial curvature we only need concern ourselves with $1 \leq i,j,k \leq 3$

Suppose i,j,k are all distinct. Then

$$2\Gamma^k_{ij} = g^{k1}(\frac{\partial g_{j1}}{\partial x^i} + \frac{\partial g_{i1}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^1}) + g^{k2}(\frac{\partial g_{j2}}{\partial x^i} + \frac{\partial g_{i2}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^2}) + g^{k3}(\frac{\partial g_{j3}}{\partial x^i} + \frac{\partial g_{i3}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^3})$$

$$2\Gamma^k_{ij} = g^{k1}(\frac{\partial g_{j1}}{\partial x^i} + \frac{\partial g_{i1}}{\partial x^j}) + g^{k2}(\frac{\partial g_{j2}}{\partial x^i} + \frac{\partial g_{i2}}{\partial x^j}) + g^{k3}(\frac{\partial g_{j3}}{\partial x^i} + \frac{\partial g_{i3}}{\partial x^j})$$

and since $g_{\mu\nu} = 0$ when $\mu \neq \nu$

$2\Gamma^1_{ij} = g^{11}(\frac{\partial g_{j1}}{\partial x^i} + \frac{\partial g_{i1}}{\partial x^j}) = 0$ since i,j,k are distinct.

$$2\Gamma^2_{ij} = g^{22}(\frac{\partial g_{j2}}{\partial x^i} + \frac{\partial g_{i2}}{\partial x^j}) = 0$$

$$2\Gamma^3_{ij} = g^{33}(\frac{\partial g_{j3}}{\partial x^i} + \frac{\partial g_{i3}}{\partial x^j}) = 0$$

So, we only need concern ourselves where at least two of the indices are the same.

Case 1: $k = i$:

$$2\Gamma^i_{ij} = g^{im}(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m})$$

$= g^{i0}(\frac{\partial g_{j0}}{\partial x^i} + \frac{\partial g_{i0}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^0}) + g^{i1}(\frac{\partial g_{j1}}{\partial x^i} + \frac{\partial g_{i1}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^1})$

$+g^{i2}(\frac{\partial g_{j2}}{\partial x^i} + \frac{\partial g_{i2}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^2}) + g^{i3}(\frac{\partial g_{j3}}{\partial x^i} + \frac{\partial g_{i3}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^3})$

$2\Gamma^1_{1j} = g^{11}(\frac{\partial g_{j1}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^j} - \frac{\partial g_{1j}}{\partial x^1})$

So,

$2\Gamma^1_{11} = g^{11}(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}) = g^{11}(\frac{\partial g_{11}}{\partial x^1})) = \kappa_R^{-1}\frac{\alpha}{R^2}$

$2\Gamma^1_{12} = 2\Gamma^1_{21} = g^{11}(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^1}) = g^{11}(\frac{\partial g_{11}}{\partial x^2}) = 0$

$2\Gamma^1_{13} = 2\Gamma^1_{31} = g^{11}(\frac{\partial g_{31}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^3} - \frac{\partial g_{13}}{\partial x^1}) = g^{11}(\frac{\partial g_{11}}{\partial x^3}) = 0$

$2\Gamma^2_{2j} = g^{22}(\frac{\partial g_{j2}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^j} - \frac{\partial g_{2j}}{\partial x^2})$

So,

$2\Gamma^2_{21} = 2\Gamma^2_{12} = g^{22}(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{21}}{\partial x^2}) = g^{22}(\frac{\partial g_{22}}{\partial x^1}) = 2/R$

$2\Gamma^2_{22} = g^{22}(\frac{\partial g_{22}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2}) = g^{22}(\frac{\partial g_{22}}{\partial x^2}) = 0$

$2\Gamma^2_{23} = 2\Gamma^2_{32} = g^{22}(\frac{\partial g_{32}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^3} - \frac{\partial g_{23}}{\partial x^2}) = g^{22}(\frac{\partial g_{22}}{\partial x^3}) = 0$

$2\Gamma^3_{3j} = g^{33}(\frac{\partial g_{j3}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^j} - \frac{\partial g_{3j}}{\partial x^3})$

So,

$2\Gamma^3_{31} = 2\Gamma^3_{13} = g^{33}(\frac{\partial g_{13}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^1} - \frac{\partial g_{31}}{\partial x^3}) = g^{33}(\frac{\partial g_{33}}{\partial x^1}) = 2R^{-1}$

$2\Gamma^3_{32} = 2\Gamma^3_{23} = g^{33}(\frac{\partial g_{23}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^2} - \frac{\partial g_{32}}{\partial x^3}) = g^{33}(\frac{\partial g_{33}}{\partial x^2}) = 2cot\theta$

$2\Gamma^3_{33} = g^{33}(\frac{\partial g_{33}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^3}) = g^{33}(\frac{\partial g_{33}}{\partial x^3}) = 0$

Case 2: $i = j$:

$2\Gamma^k_{ii} = g^{km}(\frac{\partial g_{im}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^m})$

$= g^{k0}(\frac{\partial g_{i0}}{\partial x^i} + \frac{\partial g_{i0}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^0})$

$+g^{k1}(\frac{\partial g_{i1}}{\partial x^i} + \frac{\partial g_{i1}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^1})$

$+g^{k2}(\frac{\partial g_{i2}}{\partial x^i} + \frac{\partial g_{i2}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^2})$

$+g^{k3}(\frac{\partial g_{i3}}{\partial x^i} + \frac{\partial g_{i3}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^3})$

Then,

$2\Gamma^1_{ii} = g^{11}(\frac{\partial g_{i1}}{\partial x^i} + \frac{\partial g_{i1}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^1})$

So,

$2\Gamma^1_{22} = g^{11}(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}) = -g^{11}(\frac{\partial g_{22}}{\partial x^1}) = -2\kappa_R R$

$2\Gamma^1_{33} = g^{11}(\frac{\partial g_{31}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^1}) = -g^{11}(\frac{\partial g_{33}}{\partial x^1}) = -2 sin\theta cos\theta$

$2\Gamma^2_{ii} = g^{22}(\frac{\partial g_{i2}}{\partial x^i} + \frac{\partial g_{i2}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^2})$

So,

$2\Gamma^2_{11} = g^{22}(\frac{\partial g_{12}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}) = -g^{22}(\frac{\partial g_{11}}{\partial x^2}) = 0$

$2\Gamma^2_{33} = g^{22}(\frac{\partial g_{32}}{\partial x^3} + \frac{\partial g_{32}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2}) = -g^{22}(\frac{\partial g_{33}}{\partial x^2}) = -2 sin\theta cos\theta$

$2\Gamma^3_{ii} = g^{33}(\frac{\partial g_{i3}}{\partial x^i} + \frac{\partial g_{i3}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^3})$

So,

$2\Gamma^3_{11} = g^{33}(\frac{\partial g_{13}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^3}) = -g^{33}(\frac{\partial g_{11}}{\partial x^3}) = 0$

$2\Gamma^3_{22} = g^{33}(\frac{\partial g_{23}}{\partial x^2} + \frac{\partial g_{23}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^3}) = -g^{33}(\frac{\partial g_{22}}{\partial x^3}) = 0$

Summary:

$$\Gamma^1 = \begin{pmatrix} \frac{1}{2}(1-\alpha/R)^{-1}\frac{\alpha}{R^2}R' & 0 & 0 \\ 0 & -(1-\alpha/R)RR' & 0 \\ 0 & 0 & -sin\theta cos\theta \end{pmatrix}$$

$$\Gamma^2 = \begin{pmatrix} 0 & R'/R & 0 \\ R'/R & 0 & 0 \\ 0 & 0 & -sin\theta cos\theta \end{pmatrix}$$

$$\Gamma^3 = \begin{pmatrix} 0 & 0 & R'/R \\ 0 & 0 & cot\theta \\ R'/R & cot\theta & 0 \end{pmatrix}$$

The covariant derivative is $\nabla_j \mathbf{u} = (\frac{\partial u^i}{\partial x^j} + u^k \Gamma^i_{kj})\mathbf{e}_i$

The second covariant derivative is

$$\nabla_l \nabla_j \mathbf{u} = (\frac{\partial(\frac{\partial u^i}{\partial x^j} + u^k \Gamma^i_{kj})}{\partial x^l} + (\frac{\partial u^m}{\partial x^j} + u^k \Gamma^m_{kj})\Gamma^i_{ml})\mathbf{e}_i$$

$$= (\frac{\partial^2 u^i}{\partial x^l \partial x^j} + \frac{\partial u^k}{\partial x^l}\Gamma^i_{kj} + u^k \frac{\partial}{\partial x^l}\Gamma^i_{kj} + (\frac{\partial u^m}{\partial x^j} + u^k \Gamma^m_{kj})\Gamma^i_{ml})\mathbf{e}_i$$

Then

$$[\nabla_l, \nabla_j]\mathbf{u} = (\frac{\partial u^k}{\partial x^l}\Gamma^i_{kj} + u^k \frac{\partial}{\partial x^l}\Gamma^i_{kj} + (\frac{\partial u^m}{\partial x^j} + u^k \Gamma^m_{kj})\Gamma^i_{ml}$$

$$-\frac{\partial u^k}{\partial x^j}\Gamma^i_{kl} - u^k \frac{\partial}{\partial x^j}\Gamma^i_{kl} - (\frac{\partial u^m}{\partial x^l} + u^k \Gamma^m_{kl})\Gamma^i_{mj})\mathbf{e}_i$$

$$[\nabla_l, \nabla_j]\mathbf{u} = u^k(\frac{\partial}{\partial x^l}\Gamma^i_{kj} - \frac{\partial}{\partial x^j}\Gamma^i_{kl} + \Gamma^m_{kj}\Gamma^i_{ml} - \Gamma^m_{kl}\Gamma^i_{mj})\mathbf{e}_i$$

and

$$[\nabla_1, \nabla_2]\mathbf{u} = (\frac{\partial u^k}{\partial x^1}\Gamma^i_{k2} + u^k \frac{\partial}{\partial x^1}\Gamma^i_{k2} + (\frac{\partial u^m}{\partial x^2} + u^k \Gamma^m_{k2})\Gamma^i_{m1}$$

$$-\frac{\partial u^k}{\partial x^2}\Gamma^i_{k1} - u^k \frac{\partial}{\partial x^2}\Gamma^i_{k1} - (\frac{\partial u^m}{\partial x^1} + u^k \Gamma^m_{k1})\Gamma^i_{m2})\mathbf{e}_i$$

$$= (u^k \frac{\partial}{\partial x^1}\Gamma^i_{k2} + u^k \Gamma^m_{k2}\Gamma^i_{m1} - u^k \frac{\partial}{\partial x^2}\Gamma^i_{k1} - u^k \Gamma^m_{k1}\Gamma^i_{m2})\mathbf{e}_i$$

$$= u^k(\frac{\partial}{\partial x^1}\Gamma^i_{k2} - \frac{\partial}{\partial x^2}\Gamma^i_{k1} + \Gamma^m_{k2}\Gamma^i_{m1} - \Gamma^m_{k1}\Gamma^i_{m2})\mathbf{e}_i$$

$$[\nabla_1, \nabla_3]\mathbf{u} = (\frac{\partial u^k}{\partial x^1}\Gamma^i_{k3} + u^k \frac{\partial}{\partial x^1}\Gamma^i_{k3} + (\frac{\partial u^m}{\partial x^3} + u^k \Gamma^m_{k3})\Gamma^i_{m1}$$

$$-\frac{\partial u^k}{\partial x^3}\Gamma^i_{k1} - u^k \frac{\partial}{\partial x^3}\Gamma^i_{k1} - (\frac{\partial u^m}{\partial x^1} + u^k \Gamma^m_{k1})\Gamma^i_{m3})\mathbf{e}_i$$

$$= (u^k \frac{\partial}{\partial x^1}\Gamma^i_{k3} + u^k \Gamma^m_{k3}\Gamma^i_{m1} - u^k \frac{\partial}{\partial x^3}\Gamma^i_{k1} - u^k \Gamma^m_{k1}\Gamma^i_{m3})\mathbf{e}_i$$

$$= u^k(\tfrac{\partial}{\partial x^1}\Gamma^i_{k3} - \tfrac{\partial}{\partial x^3}\Gamma^i_{k1} + \Gamma^m_{k3}\Gamma^i_{m1} - \Gamma^m_{k1}\Gamma^i_{m3})\mathbf{e}_i$$

$$[\nabla_2, \nabla_3]\mathbf{u} = (\tfrac{\partial u^k}{\partial x^2}\Gamma^i_{k3} + u^k\tfrac{\partial}{\partial x^2}\Gamma^i_{k3} + (\tfrac{\partial u^m}{\partial x^3} + u^k\Gamma^m_{k3})\Gamma^i_{m2}$$

$$-\tfrac{\partial u^k}{\partial x^3}\Gamma^i_{k2} - u^k\tfrac{\partial}{\partial x^3}\Gamma^i_{k2} - (\tfrac{\partial u^m}{\partial x^2} + u^k\Gamma^m_{k2})\Gamma^i_{m3})\mathbf{e}_i$$

$$= (u^k\tfrac{\partial}{\partial x^2}\Gamma^i_{k3} + u^k\Gamma^m_{k3}\Gamma^i_{m2} - u^k\tfrac{\partial}{\partial x^3}\Gamma^i_{k2} - u^k\Gamma^m_{k2}\Gamma^i_{m3})\mathbf{e}_i$$

$$= u^k(\tfrac{\partial}{\partial x^2}\Gamma^i_{k3} - \tfrac{\partial}{\partial x^3}\Gamma^i_{k2} + \Gamma^m_{k3}\Gamma^i_{m2} - \Gamma^m_{k2}\Gamma^i_{m3})\mathbf{e}_i$$

We assume unit basis vectors in the direction of the orthogonal coordinates r, θ, ϕ where $R = (r^3 + \alpha^3)^{1/3}$ and compute the components of the scalar curvature where

$$S = 2\Sigma_{i<j}\langle[\nabla_i, \nabla_j]\mathbf{e}_i, \mathbf{e}_j\rangle$$

$$\langle[\nabla_1, \nabla_2]\mathbf{e}_1, \mathbf{e}_2\rangle = \langle(\tfrac{\partial}{\partial x^1}\Gamma^i_{22} - \tfrac{\partial}{\partial x^2}\Gamma^i_{21} + \Gamma^m_{22}\Gamma^i_{m1} - \Gamma^m_{21}\Gamma^i_{m2})\mathbf{e}_i, \mathbf{e}_2\rangle$$

$$= (\tfrac{\partial}{\partial x^1}\Gamma^2_{22} - \tfrac{\partial}{\partial x^2}\Gamma^2_{21} + \Gamma^m_{22}\Gamma^2_{m1} - \Gamma^m_{21}\Gamma^2_{m2})\langle\mathbf{e}_2, \mathbf{e}_2\rangle$$

$$= (\tfrac{\partial}{\partial x^1}\Gamma^2_{22} - \tfrac{\partial}{\partial x^2}\Gamma^2_{21} + \Gamma^m_{22}\Gamma^2_{m1} - \Gamma^m_{21}\Gamma^2_{m2}) = 0$$

$$\langle[\nabla_1, \nabla_3]\mathbf{e}_1, \mathbf{e}_3\rangle = \langle(\tfrac{\partial}{\partial x^1}\Gamma^i_{33} - \tfrac{\partial}{\partial x^3}\Gamma^i_{31} + \Gamma^m_{33}\Gamma^i_{m1} - \Gamma^m_{31}\Gamma^i_{m3})\mathbf{e}_i, \mathbf{e}_3\rangle$$

$$= (\tfrac{\partial}{\partial x^1}\Gamma^3_{33} - \tfrac{\partial}{\partial x^3}\Gamma^3_{31} + \Gamma^m_{33}\Gamma^3_{m1} - \Gamma^m_{31}\Gamma^3_{m3})\langle\mathbf{e}_3, \mathbf{e}_3\rangle$$

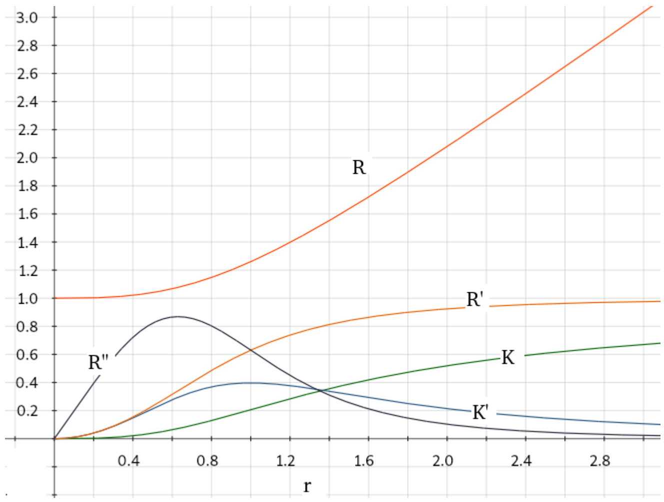
$$= (\tfrac{\partial}{\partial x^1}\Gamma^3_{33} - \tfrac{\partial}{\partial x^3}\Gamma^3_{31} + \Gamma^m_{33}\Gamma^3_{m1} - \Gamma^m_{31}\Gamma^3_{m3}) = 0$$

$$\langle[\nabla_2, \nabla_3]\mathbf{e}_2, \mathbf{e}_3\rangle = \langle(\tfrac{\partial}{\partial x^2}\Gamma^i_{33} - \tfrac{\partial}{\partial x^3}\Gamma^i_{32} + \Gamma^m_{33}\Gamma^i_{m2} - \Gamma^m_{32}\Gamma^i_{m3})\mathbf{e}_i, \mathbf{e}_3\rangle$$

$$= (\tfrac{\partial}{\partial x^2}\Gamma^3_{33} - \tfrac{\partial}{\partial x^3}\Gamma^3_{32} + \Gamma^m_{33}\Gamma^3_{m2} - \Gamma^m_{32}\Gamma^3_{m3})\langle\mathbf{e}_3, \mathbf{e}_3\rangle$$

$$= (\tfrac{\partial}{\partial x^2}\Gamma^3_{33} - \tfrac{\partial}{\partial x^3}\Gamma^3_{32} + \Gamma^m_{33}\Gamma^3_{m2} - \Gamma^m_{32}\Gamma^3_{m3}) = 0$$

Therefore $S = 0$. We can plot the circular symmetric elements in the figure below (at $\alpha = 1$): $\kappa = (1 - \frac{\alpha}{R})$ and derivatives are w.r.t. r.



As discussed above each member of the foliation of "hyperboloids" becomes flat near $r = 0$.

At $r = 0$,
For the 1st fundamental form:

$$\begin{aligned} E &= (r^3+\alpha^3)^{2/3} = \alpha^2 \\ F &= 0 \\ G &= \frac{r^4}{(r^3+\alpha^3)^{4/3}} + 1 = 1 \end{aligned}$$

and the 2nd fundamental form:

$$\begin{aligned} e &= -\frac{(r^3+\alpha^3)^{1/3}}{\sqrt{\frac{r^4}{(r^3+\alpha^3)^{4/3}}+1}} = -\alpha \\ f &= 0 \\ g &= \frac{2r(r^3+\alpha^3)^{2/3} - 3r^4(r^3+\alpha^3)^{-1/3}}{(r^3+\alpha^3)^{4/3}} \Big/ \sqrt{\frac{r^4}{(r^3+\alpha^3)^{4/3}}+1} = 0 \end{aligned}$$

$\kappa_1 = g/G = 0$ and $\kappa_2 = e/E = -1/\alpha$ so $\kappa_1\kappa_2 = 0$.

Discussion: The full curvature tensor is given by

$$[\nabla_X, \nabla_Y](Z) - \nabla_{[X,Y]}(Z)$$

where X, Y, Z are vector fields. When X and Y are elements of an orthogonal frame field this reduces to

$$[\nabla_i, \nabla_j](Z)$$

as employed above giving $S = 2\Sigma_{i<j}\langle[\nabla_i, \nabla_j]\mathbf{e}_i, \mathbf{e}_j\rangle$

The curvature of the "hyperboloids" (non-zero away from $r = 0$) captures the rate of divergence of the gravitational flux compared to flat space. Clearly the curvature tensor does not. The ratio of gravitational flux

$$= \frac{A_r}{A_R} = \frac{4\pi r^2}{4\pi R^2} = \frac{r^2}{R^2}$$

Consider the example of 3-D space in spherical coordinates with a metric $ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$. It is flat. However, restricting to a sphere of given radius $r = a$, that sphere is curved with metric

$$d\sigma^2 = a^2(d\theta^2 + sin^2\theta d\phi^2)$$

We can consider all such spheres as a foliation indexed by a. This analysis becomes of crucial importance in the discussion of rotating Black Holes. See the article *The Kerr Metric.*

*** Footnote: Derivation of Energy Equation:

The geodesic equations

$\frac{d}{ds}[g_{ii}\frac{dx^i}{ds}] = \frac{1}{2}\Sigma_{j=0}^3 \partial_i g_{jj}\frac{dx^j}{ds}\frac{dx^j}{ds}$ for $0 \leq i \leq 3$ give the geodesic equations of motion with respect to the arclength parameter s.

The solution of the field equations of General Relativity was given by Karl Schwarzschild in 1916 who derived the spherically symmetric solution

$ds^2 = (1 - \frac{\alpha}{R})dt^2 - (1 - \frac{\alpha}{R})^{-1}dR^2 - R^2(d\theta^2 + sin^2\theta d\phi^2)$

where t is the time 'at infinity', $R = (r^3 + \alpha^3)^{1/3}$ is the area radius, and $\alpha = 2GM$.

$g_{00} = 1 - \frac{\alpha}{R}$, $g_{11} = (1 - \frac{\alpha}{R})^{-1}$, $g_{22} = R^2$, and $g_{33} = R^2 sin^2\theta$.

$\partial_0 g_{jj} = 0$ for all j so $g_{00}\frac{dx^0}{ds} = constant = E$ where E is the total energy and can be identified as the Hamiltonian of the system.

$\partial_3 g_{jj} = 0$ for all j so $g_{33}\frac{dx^3}{ds} = constant = L$ where L is the angular momentum.

The gravitational field is spherically symmetric so we can transform the angle coordinates to our convenience. So, letting $\theta = \pi/2$ we have

$g_{33}\frac{dx^3}{ds} = R^2\frac{d\phi}{ds} = L$

For convenience in what follows let $\kappa_R = 1 - \frac{\alpha}{R}$

For a photon following a geodesic path

$\frac{d\gamma}{ds} = \frac{dt}{ds}\partial_t + \frac{dR}{ds}\partial_R + \frac{d\theta}{ds}\partial_\theta + \frac{d\phi}{ds}\partial_\phi$

$ds^2 = (1 - \frac{\alpha}{R})dt^2 - (1 - \frac{\alpha}{R})^{-1}dR^2 - R^2(d\theta^2 + sin^2\theta d\phi^2)$

The inner product $0 = \langle\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\rangle$

$= (\frac{dt}{ds})^2\langle\partial_t, \partial_t\rangle + (\frac{dR}{ds})^2\langle\partial_R, \partial_R\rangle + (\frac{d\theta}{ds})^2\langle\partial_\theta, \partial_\theta\rangle + (\frac{d\phi}{ds})^2\langle\partial_\phi, \partial_\phi\rangle$

$= (E/\kappa_R)^2\kappa_R - (\frac{dR}{ds})^2\kappa_R^{-1} - (\frac{d\phi}{ds})^2 R^2$

$= E^2\kappa_R^{-1} - (\frac{dR}{ds})^2\kappa_R^{-1} - (\frac{L}{R^2})^2 R^2 = E^2\kappa_R^{-1} - (\frac{dR}{ds})^2\kappa_R^{-1} - \frac{L^2}{R^2}$

Then

$E^2\kappa_R^{-1} = (\frac{dR}{ds})^2\kappa_R^{-1} + \frac{L^2}{R^2}$

$E^2 = (\frac{dR}{ds})^2 + \kappa_R\frac{L^2}{R^2}$

So, the *energy equation* is:

$E^2 = (\frac{dR}{ds})^2 + \kappa_R \frac{L^2}{R^2}$

To compute the energy equation for a material particle of unit mass we replace s with proper time τ and then $1 = \langle \frac{d\gamma}{d\tau}, \frac{d\gamma}{d\tau} \rangle$ giving the energy equation

$E^2 = (\frac{dR}{d\tau})^2 + \kappa_R (1 + \frac{L^2}{R^2})$

Appendix:

Local Isometries:

Consider first polar coordinates in the plane. With r as the radial length and θ as the angle a radial line makes relative to the horizontal we can deduce that the metric will be $ds^2 = dr^2 + r^2 d\theta^2$. Now consider the transformation:

$$\begin{aligned} x &= rcos\theta \\ y &= rsin\theta \end{aligned}$$

That is, $(x, y) = \Phi(r, \theta) = (rcos\theta, rsin\theta)$

Then

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = D\Phi \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

and

$$dx^2 + dy^2 = \begin{pmatrix} dx \\ dy \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = [D\Phi \begin{pmatrix} dr \\ d\theta \end{pmatrix}]^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} [D\Phi \begin{pmatrix} dr \\ d\theta \end{pmatrix}]$$

Then

$$dx^2 + dy^2 = \begin{pmatrix} dr \\ d\theta \end{pmatrix}^T D\Phi^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D\Phi \begin{pmatrix} dr \\ d\theta \end{pmatrix} = \begin{pmatrix} dr \\ d\theta \end{pmatrix}^T (D\Phi^T)(D\Phi) \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

Then

$$dx^2 + dy^2 = \begin{pmatrix} dr \\ d\theta \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = dr^2 + r^2 d\theta^2$$

Therefore Φ is a local isometry.

Now consider spherical coordinates. With r as the radial length and θ and ϕ corresponding to latitude and longitude we can deduce that the metric will be $ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$. Now consider the transformation:

$$\begin{aligned} x &= rsin\theta cos\phi \\ y &= rsin\theta sin\phi \\ z &= rcos\theta \end{aligned}$$

That is, $(x, y, z) = \Phi(r, \theta, \phi) = (rsin\theta cos\phi, rsin\theta sin\phi, rcos\theta)$

Then

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} sin\theta cos\phi & rcos\theta cos\phi & -rsin\theta sin\phi \\ sin\theta sin\phi & rcos\theta sin\phi & rsin\theta cos\phi \\ cos\theta & -rsin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = D\Phi \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

and

$$dx^2 + dy^2 + dz^2 = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$= [D\Phi \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}]^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [D\Phi \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}]$$

Then

$$dx^2 + dy^2 + dz^2 = \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}^T (D\Phi)^T (D\Phi) \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

Then

$$dx^2+dy^2+dz^2 = \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = dr^2+r^2(d\theta^2+sin^2\theta d\phi^2)$$

Therefore Φ is a local isometry.

Applying this to vectors we have:
First for polar coordinates:

$$(\overline{\mathbf{x}}^1)^2 + (\overline{\mathbf{x}}^2)^2 = (dx^2 + dy^2)(\overline{\mathbf{x}}, \overline{\mathbf{x}})$$

and

$$(dx^2 + dy^2)(\overline{\mathbf{x}}, \overline{\mathbf{x}}) = \begin{pmatrix} \overline{\mathbf{x}}^1 \\ \overline{\mathbf{x}}^2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}}^1 \\ \overline{\mathbf{x}}^2 \end{pmatrix}$$

$$= [D\Phi \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{pmatrix}]^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} [D\Phi \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{pmatrix}]$$

Then

$$(dx^2 + dy^2)(\overline{\mathbf{x}}, \overline{\mathbf{x}}) = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{pmatrix}^T (D\Phi)^T (D\Phi) \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{pmatrix}$$

Then

$$(dx^2 + dy^2)(\overline{\mathbf{x}}, \overline{\mathbf{x}}) = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{pmatrix} = (\mathbf{x}^1)^2 + r^2(\mathbf{x}^2)^2$$

$$= (dr^2 + r^2 d\theta^2)(\mathbf{x}, \mathbf{x})$$

And for spherical coordinates:

$$(\overline{\mathbf{x}}^1)^2 + (\overline{\mathbf{x}}^2)^2 + (\overline{\mathbf{x}}^3)^2 = (dx^2 + dy^2 + dz^2)(\overline{\mathbf{x}}, \overline{\mathbf{x}})$$

and

$$(dx^2 + dy^2 + dz^2)(\overline{\mathbf{x}}, \overline{\mathbf{x}}) = \begin{pmatrix} \overline{\mathbf{x}}^1 \\ \overline{\mathbf{x}}^2 \\ \overline{\mathbf{x}}^3 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{x}}^1 \\ \overline{\mathbf{x}}^2 \\ \overline{\mathbf{x}}^3 \end{pmatrix}$$

$$= [D\Phi \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{pmatrix}]^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [D\Phi \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{pmatrix}]$$

Then

$$(dx^2 + dy^2 + dz^2)(\overline{\mathbf{x}}, \overline{\mathbf{x}}) = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{pmatrix}^T (D\Phi)^T (D\Phi) \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{pmatrix}$$

Then

$$(dx^2+dy^2+dz^2)(\overline{\mathbf{x}}, \overline{\mathbf{x}}) = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{pmatrix} \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{pmatrix} = (\mathbf{x}^1)^2 + r^2((\mathbf{x}^2)^2 + sin^2\theta(\mathbf{x}^3)^2)$$

$$= (dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2))(\mathbf{x}, \mathbf{x})$$

Is the transformation from $r \to R$ a local isometry?

Let $(t, R, \theta, \phi) = \Phi(t, r, \theta, \phi) = (t, (r^3 + \alpha^3)^{1/3}, \theta, \phi)$

$$\begin{aligned} \partial_t t &= 1 \\ \partial_t R &= 0 \end{aligned}$$

$$
\begin{aligned}
\partial_t \theta &= 0 \\
\partial_t \phi &= 0 \\
\partial_r t &= 0 \\
\partial_r R &= (1/3)(r^3 + \alpha^3)^{-2/3}(3r^2) \\
\partial_r \theta &= 0 \\
\partial_r \phi &= 0 \\
\partial_\theta t &= 0 \\
\partial_\theta R &= 0 \\
\partial_\theta \theta &= 1 \\
\partial_\theta \phi &= 0 \\
\partial_\phi t &= 0 \\
\partial_\phi R &= 0 \\
\partial_\phi \theta &= 0 \\
\partial_\phi \phi &= 1
\end{aligned}
$$

Then:

$$
D\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_r R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Set

$$
g = \begin{pmatrix} 1 - \alpha/r & 0 & 0 & 0 \\ 0 & (1 - \alpha/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}
$$

and set

$$
\overline{g} = \begin{pmatrix} 1 - \alpha/R & 0 & 0 & 0 \\ 0 & (1 - \alpha/R)^{-1} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2 sin^2\theta \end{pmatrix}
$$

Let $\overline{\mathbf{x}}$ be a vector w.r.t. the basis $(\mathbf{e}_t, \mathbf{e}_R, \mathbf{e}_\theta, \mathbf{e}_\phi,)$ where $\mathbf{e}_\sigma = \frac{\partial}{\partial\sigma}/|\frac{\partial}{\partial\sigma}|$ and $\mathbf{x}$ be the same vector w.r.t the basis $(\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi,)$. The vectors are assumed to be carrying a Minkowski signature. Then,

$$\mathbf{x} = \begin{pmatrix} x^0 \\ ix^1 \\ ix^2 \\ ix^3 \end{pmatrix} \text{ and } \overline{\mathbf{x}} = D\Phi\mathbf{x}$$

Let $< \mathbf{x}, \mathbf{x} >_g$ be the metric represented by g and $< \overline{\mathbf{x}}, \overline{\mathbf{x}} >_{\overline{g}}$ be the metric represented by $\overline{g}$. Then

$$< \overline{\mathbf{x}}, \overline{\mathbf{x}} >_{\overline{g}} = (\overline{\mathbf{x}})^T \overline{g}(\overline{\mathbf{x}}) = (D\Phi\mathbf{x})^T \overline{g}(D\Phi\mathbf{x}) = \mathbf{x}^T (D\Phi)^T \overline{g}(D\Phi)\mathbf{x} = < \mathbf{x}, \mathbf{x} >_{(D\Phi)^T \overline{g}(D\Phi)}$$

But $(D\Phi)^T \overline{g}(D\Phi) \neq g$ unless $r = R$ which is not the case. So, $D\Phi$ is not a local isometry and therefore $< \overline{\mathbf{x}}, \overline{\mathbf{x}} >_{\overline{g}} \neq < \mathbf{x}, \mathbf{x} >_g$. Therefore g and $\overline{g}$ do not represent the same metric. We did not need to plug in the specific expression for $\partial_r R$ so we can say that any $\Phi(t, r, \theta, \phi) = (t, f(r), \theta, \phi)$ produces a different metric than $\overline{g}$ if $f(r) \neq (r^3 + \alpha^3)^{1/3}$.

Now, using $\overline{g}$ in what follows we see that letting $r = \alpha$ we have:

$$R = (r^3 + \alpha^3)^{1/3} = (2\alpha^3)^{1/3} = \sqrt[3]{2}\alpha$$

$$ds^2 = (1 - 1/\sqrt[3]{2})dt^2 - (1 - 1/\sqrt[3]{2})^{-1} dR^2 - 4(2)^{2/3} d\Omega^2$$

So, time slows down by a factor of $1 - 1/\sqrt[3]{2}$ compared to a clock 'at infinity' and radial length contracts by the same factor. But, there is no coordinate singularity at $r = \alpha$ meaning time does not slow down and stop there relative to a clock 'at infinity'.

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