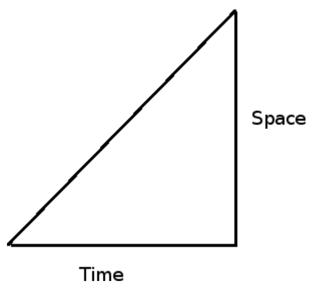
A Simplified Version of the Theory

The Theory of General Relativity shows the space and time are intimately linked in what is often called space-time. The linkage might suggest that without space there is no time. The theory we have developed shows that time and space are complementary opposites to each other. They are related in a kind of Pythagorean triangle.



How do we calculate the length of the hypotenuse, the third line in the triangle that is neither space nor time but a mixture of the two. This is where Special Relativity comes in. The obvious thing to do would be to apply the Pythagoras Theorem and get $H^2 = T^2 + S^2$ where H, T, and S are the lengths of the Hypotenuse, Time, and Space respectively. In this case the obvious approach turns out to be wrong but not entirely. It is wrong because such a computation does not preserve what is called Lorentz invariance which is required due to the fact that all observers must measure H to be the same regardless of their (uniform) state of motion. However, if we view our triangle as being inside a complex plane we get Lorenz invariance.

The complex plane measures horizontal distances in the usual way but vertical distances are multiplied by the number i. This number has the rather unusual property that $i^2 = -1$. That is, $i \times i = -1$. This property is so unusual that originally these numbers were thought to be only imaginary. Though we do not regard them that way any longer the name stuck so they are still called imaginary numbers.

How does this help us with our triangle? We measure Space in the imaginary direction and replace S with iS. Then we express our new formula as $H^2 = T^2 + (iS)^2 = H^2 = T^2 - S^2$. This is the correct way to express the length of the hypotenuse that is consistent with the requirement of Lorentz invariance. This creates a bit of a paradox. Can H be zero while T and S

are both non-zero? Yes, if $T = S$. If we express distances in light units, for example one light-year or one light-second, meaning the distance light travels in a year or a second respectively, then light travels one light-second in a second. Then for light $T = S$ so the hypotenuse is always zero for light. For everything traveling at less that light speed, $H^2 = T^2 - S^2$ is greater than zero.

So far, we have looked at space as being one dimensional. Actually space is three dimensional with length, width, and height. How do we measure H now? We generalize a bit and introduce three "imaginary" numbers $\mathbf{i}, \mathbf{j}, \mathbf{k}$ called quaternions. The set $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ then form a 4-dimensional system of vectors. The quaternions have an additional property that their multiplication is order dependent. That is, $\mathbf{ij} = -\mathbf{ji}$, $\mathbf{ik} = -\mathbf{ki}$, and $\mathbf{kj} = -\mathbf{jk}$. We call this the anti-commutative property and it turns out to be very useful. So, our new formula becomes $H^2 = T^2 + (\mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z)^2$, where X, Y, Z represent our three spatial dimensions of length width and height. Now, when we calculate $(\mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z)^2$ we get something very interesting. The $\mathbf{i}, \mathbf{j}, \mathbf{k}$ act like imaginary numbers and the anti-commutative property ensures that the cross terms all cancel out so we get $(\mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z)^2 = -X^2 - Y^2 - Z^2$ and our new improved formula becomes $H^2 = T^2 - X^2 - Y^2 - Z^2$.

Lorentz invariance was referred to above. It is important for our story so let's examine it a bit more. Suppose there are two travelers moving at a constant speed relative to each other and they both witness an event. For our purposes we will say an event is a space-time interval. That is, it has an elapsed time and interval in space. We will use the notation Δ to indicate the difference between the initial value and final value of an interval. So, $\Delta T = T(final) - T(initial)$, $\Delta X = X(final) - X(initial)$ etc. and then our formula becomes $\Delta H^2 = \Delta T^2 - \Delta X^2 - \Delta Y^2 - \Delta Z^2$. We must distinguish the space-time interval for the two travelers by giving one a prime symbol. So, for the first, $\Delta H^2 = \Delta T^2 - \Delta X^2 - \Delta Y^2 - \Delta Z^2$ and for the second, $\Delta H'^2 = \Delta T'^2 - \Delta X'^2 - \Delta Y'^2 - \Delta Z'^2$. Lorentz invariance requires that $\Delta H^2 = (\Delta H')^2$.

Lorentz Invariance

$$\begin{pmatrix} \Delta T' \\ \Delta X' \\ \Delta Y' \\ \Delta Z' \end{pmatrix} = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta T \\ \Delta X \\ \Delta Y \\ \Delta Z \end{pmatrix}$$

In this case the Lorentz transformation

$$\Lambda = \begin{pmatrix} cosh(\alpha) & sinh(\alpha) & 0 & 0 \\ sinh(\alpha) & cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

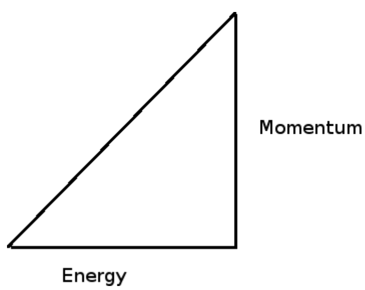
Lorentz invariance of a vector $\mathbf{v}$ can be described by $(\Lambda \mathbf{v})^T g \Lambda \mathbf{v} = \mathbf{v}^T g \mathbf{v}$

That is, $\Lambda^T g \Lambda = g$ defines a Lorentz transformation.

where $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the Minkowski metric.

Energy-Momentum

Just as space and time are complementary opposites and related in a Pythagorean triangle so also are energy and momentum.



How do we find the hypotenuse in this case and what could it possibly mean? We proceed as with space and time noting that momentum is a directional (vector) quantity whereas energy is a scalar quantity. We use the formula $H^2 = T^2 + (\mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z)^2$ but replace the spatial entries with the corresponding momenta P_X, P_Y, and P_Z and the time entry with the energy (divided by c, the speed of light). Then $H^2 = (E/c)^2 + (\mathbf{i}P_X + \mathbf{j}P_Y + \mathbf{k}P_Z)^2$. What can the H^2 possibly be? It turns out that $H = mc$, the mass of an object times the speed of light. The formula $(mc)^2 = (E/c)^2 + (\mathbf{i}P_X + \mathbf{j}P_Y + \mathbf{k}P_Z)^2$ is also Lorentz invariant. That is, for two observers in uniform motion relative to each other, they observe

$(mc)^2 = (E/c)^2 + (\mathbf{i}P_X + \mathbf{j}P_Y + \mathbf{k}P_Z)^2 = (E'/c)^2 + (\mathbf{i}P_{X'} + \mathbf{j}P_{Y'} + \mathbf{k}P_{Z'})^2$ as an invariant.

The space-time example and the energy-momentum example both represent 4-vectors given by $\mathbf{V} = (V_0, V_1, V_2, V_3)$ and in each case the magnitude $V_0^2 + (\mathbf{i}V_1 + \mathbf{j}V_2 + \mathbf{k}V_3)^2$ is Lorentz invariant. Lorentz invariance is a fact of Nature insomuch as no violation of it has ever been observed. Its broader implication is that the laws of physics are invariant with respect to uniform motion of observers. The situation where the observers' motion is non-uniform is covered by a more general theory called, appropriately enough, The General Theory of Relativity. We saw above that 4-dimensional space-time can be represented by the quaternion set $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Relativity theory describes how Nature preserves the quaternion structure of 4-vectors in all its operations.

This leads us to the main part of our story. We have seen how the quaternion structure of space-time is preserved in the operations of Nature. The Quaternions are referred to in Mathematics as a division algebra. They form the largest division algebra whose multiplication is associative, meaning $a(bc) = (ab)c$. Just as we can form an exponential of a real number $e^x = exp(x)$ we can also form the exponential of a quaternion $exp(\mathbf{1}\tau + \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z)$. When we do that to all the quaternions at once we get a set of values $\mathbf{R}^+ \times S^3$. The first term $\mathbf{R}^+$ represents the set of positive real numbers (i.e. excluding zero). S^3 represents the three dimensional sphere. It is not to be thought of as a three dimensional solid ball. It is like the surface of the ball but raised one dimension higher. The product $\times$ is the *Cartesian product* of sets. The expression $\mathbf{R}^+ \times S^3$ can be used to model an expanding three dimensional spherical Universe. An important point to notice is that $\mathbf{R}^+$ represents the time component so $t > 0$. Notice that when we formed the exponential of the quaternions we wrote $exp(\mathbf{1}\tau + \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z)$. The τ value is the coefficient of the quaternion vector $\mathbf{1}$.

We can speak of the Universe beginning as a quantum fluctuation. Consider the Heisenberg relation: $\sigma_x \sigma_y \geq \hbar/2$ where x and y are conjugate variables and σ is the standard deviation of the uncertainty. Energy (E) and time (t) are such conjugates and so $\sigma_E \sigma_t \geq \hbar/2$. As time becomes more constrained, uncertainty in the energy grows accordingly. In the limit as $\sigma_t \to 0$, $\sigma_E \to \infty$. Consequently, we cannot assign a value $t = 0$ to time.

There is one problem here. The Universe is not a 3-sphere. However, given the facts contained in this website the 3-sphere can be replaced with a 3-pseudosphere S^3_P. In the above exponential we replace the quaternions with bi-quaternions. These are the quaternions with each multiplied by $i = \sqrt{-1}$. So, the Universe expanded as a 3-pseudosphere with the topology $\mathbf{R}^+ \times S^3_P$.